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BROWNIAN MOTION ON THE LINE (II)*

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§4. Drift

The methods above can be used to obtain analogous results for a Brownian motion with a constant *drift*, namely for the process:

$$(1) \qquad \tilde{X}(t) = X(t) + ct$$

where $X(t)$ is the standard Brownian motion and c is a nonzero constant. We may suppose $c>0$ for definiteness.

The strong law of large numbers implies that almost surely

$$(2) \qquad \lim_{t\to\infty} \tilde{X}(t) = +\infty.$$

The argument in §1 is still valid to show that exit from any given interval (a,b) is almost sure, but the analogue to Proposition 1 must be false. The reader should find out for himself that the martingales in (3) and (10) of §2, translated in terms of $\tilde{X}$, are not sufficient to determine

$$(3) \qquad \tilde{p}_a(X) = P^x\{\tilde{X}(\tilde{\tau}) = a\}, \qquad \tilde{p}_b(X) = P^x\{\tilde{X}(\tilde{\tau}) = b\},$$

where

$$\tilde{\tau} = \tilde{\tau}_{(a,b)} = \inf\{t>0: \tilde{X}(t) \notin (a,b)\}.$$

Fortunately, the martingale in (1) of §3 can be manipulated to do so.

Take $a = -2c$ in (1) of §3. We have

$$(4) \qquad \exp(-2cX(t) - \frac{(2c)^2 t}{2}) = \exp(-2c\tilde{X}(t)).$$

Write $s(x) = e^{-2cx}$, then

$$(5) \qquad \{s(\tilde{X}(t)), F_t, P^x\} \quad \text{is a martingale.}$$

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It follows that

(6) $$s(x)=s(a)\tilde{p}_a(x)+s(b)\tilde{p}_b(x)$$

together with

(7) $$\tilde{p}_a(x)+\tilde{p}_b(x)=1,$$

we obtain

(8) $$\tilde{p}_a(x)=\frac{s(b)-s(x)}{s(b)-s(a)},\quad \tilde{p}_b(x)=\frac{s(x)-s(a)}{s(b)-s(a)}.$$

The function s is called the **scale** function. Compare (8) with (9) of §1. We can now use the martingale $\tilde{X}(t)-ct$ to find $E^x\{\tilde{\tau}\}$. As before, the stopping theorem yields

$$x=E^x\{\tilde{X}(\tilde{\tau})-c\tilde{\tau}\}=a\tilde{p}_a(x)+b\tilde{p}_b(x)-cE^x\{\tilde{\tau}\},$$

and so

$$E^x\{\tilde{\tau}\}=\frac{a(s(b)-s(x))+b(s(x)-s(a))-x(s(b)-s(a))}{c(s(b)-s(a))}.$$

More interesting is to use (8) to obtain information for the hitting time

$$T_y=\inf\{t>0: X(t)=y\}.$$

If we let $b\to\infty$ in the first and $a\to-\infty$ in the second equation in (8), the results are as follows:

(9) $$P^x\{\tilde{T}_a<\infty\}=e^{-2c(x-a)},\quad a<x<\infty;$$
$$P^x\{\tilde{T}_b<\infty\}=1,\quad -\infty<x<b.$$

The second relation is of course an immediate consequence of (2).

To obtain the distribution of $\tilde{T}_y$, we return to the martingale in (1) of §3, translated in terms of $\tilde{X}(t)$;

$$\exp(\alpha\tilde{X}(t)-\left(\alpha c+\frac{\alpha^2}{2}\right)t).$$

Put

(10) $$\lambda=\alpha c+\frac{\alpha^2}{2},$$
$$\alpha=-c\pm\sqrt{2\lambda+c^2}.$$

we obtain in the usual manner

(11) $$e^{\alpha x}=e^{\alpha a}E^x\{\exp(-\lambda\tilde{\tau});\tilde{X}(\tilde{\tau})=a\}+e^{\alpha b}E^x\{\exp(-\lambda\tilde{\tau});\tilde{X}(\tilde{\tau})=b\}.$$

Choose the + sign in α so that $\alpha>0$, and let $a\to-\infty$. Then choose the − sign in α so that $\alpha<0$, and let $b\to+\infty$. The results may be recorded as follows:

(12) $$E^x\{\exp(-\lambda\tilde{T}_y)\}=\exp(-\sqrt{2\lambda+c^2}\,|x-y|-c(x-y)).$$

Using equation (24) of §3 we can obtain the joint distribution of $\tilde{X}(\tau)$ and $\tilde{\tau}$. In general the results are complicated but one interesting case emerges when $x=0$, $b>0$ and $a=-b$. In this case if we let $f_+(\lambda)=E^0(\exp(-\lambda\tilde{\tau});\tilde{X}(\tilde{\tau})=b)$ and $f_-(\lambda)=E^0(\exp(-\lambda\tilde{\tau});\tilde{X}(\tilde{\tau})=-b)$, then (24) of §3 becomes

$$\exp(-\sqrt{2\lambda+c^2}\,b+bc)=f_+(\lambda)+f_-(\lambda)\exp(-\sqrt{2\lambda+c^2}\,(2b)+2bc),$$
$$\exp(-\sqrt{2\lambda+c^2}\,b-bc)=f_-(\lambda)+f_+(\lambda)\exp(-\sqrt{2\lambda+c^2}\,(2b)-2bc).$$

Dividing each equation by its left hand side and subtracting, we obtain

$$(13)\qquad f_+(\lambda)(\exp(\sqrt{2\lambda+c^2}\,b-bc)-\exp(-\sqrt{2\lambda+c^2}\,b-bc))$$
$$=f_-(\lambda)(\exp(\sqrt{2\lambda-c^2}\,b+bc)-\exp(-\sqrt{2\lambda+c^2}\,b+bc)),$$

and consequently

$$(14)\qquad f_+(\lambda)=e^{2bc}f_-(\lambda).$$

Since we have also from (8)

$$(15)\qquad P^0(\tilde{X}(\tilde{\tau})=b)=e^{2bc}P^0(\tilde{X}(\tilde{\tau})=-b),$$

it follows that

$$E^0(\exp(-\lambda\tilde{\tau})\,|\,\tilde{X}(\tilde{\tau})=b)=E^0(\exp(\lambda\tilde{\tau})\,|\,\tilde{X}(\tilde{\tau})=-b).$$

That is, the exit time $\tilde{\tau}$ and the exit place $\tilde{X}(\tilde{\tau})$ are independent. This curious fact was first observed by Frederick Stein.* Is there an intuitive explanation?

Exercise 12. Almost every sample function $\tilde{X}(\cdot,\omega)$ has a minimum value $m(\omega)>-\infty$. Use the strong Markov property to show that m has an exponential distribution, and then find this distribution.

Exercise 13. Show that almost every path of $\tilde{X}$ reaches its minimum value $m(\omega)$ only once.

Exercise 14. This exercise, which is based on a result of J. W. Pitman and J. C. Rogers, shows that sometimes processes which are "obviously" not Markovian actually are. Let X^+ and X^- be independent Brownian motions with drifts $+c$ and $-c$ respectively and let ξ be an independent random variable which $=+1$ with probability p and $=-1$ with probability $1-p$. Construct a process Y by letting $Y_t=X_t^+$ on $\{\xi=1\}$ and $Y_t=X_t^-$ on $\{\xi=-1\}$. The claim is that Y is a Markov process with respect to g_t, the σ-field generated by Y_s, $s\leqslant t$.

At first glance this seems false because watching Y_t gives us information about ξ which can be used to predict the future development of the process. This is true but a little more thought shows

$$P^0(\xi=1\,|\,g_t)=e^{cx_t}/(e^{cx_t}+e^{-cx_t}).$$

Verify this by (9) and show that Y_t is Markovian. [I owe this exercise to R. Durrett.]

§5. Dirichlet and Poisson Problems

In classical potential theory (see Kellogg [1]) there are a clutch of famous problems which had their origins in electromagnetism. We begin by stating two of these problems in Euclidean space R^d, where d is the dimension. Let D be a nonempty bounded open set (called a "domain" when it is connected), and let ∂D

* "An independence in Brownian motion with constant drift", *Ann. of Prob.* 5 (1977), 571-572.

denote its boundary: $\partial D = \bar{D} \cap \overline{(D^c)}$ where the upper bar denotes closure. Let $\triangle$ denote the Laplacian, namely the differential operator

$$(1) \qquad \Delta = \sum_{j=1}^{d} \left(\frac{\partial}{\partial x_j} \right)^2 .$$

A function defined in D is called *harmonic* there iff $\Delta f = 0$ in D. This of course requires that f is twice differentiable. If f is locally integrable in D, namely has a finite Lebesgue integral over any compact subset of D, then it is harmonic in D if and only if the following "surface averaging property" is true. Let $B(x,\delta)$ denote the closed ball with center x and radius δ. For each $x \in D$ and $\delta > 0$ such that $B(x,\delta) \subset D$, we have

$$(2) \qquad f(x) = \frac{1}{\sigma(\partial B(x,\delta))} \int_{\partial B(x,\delta)} f(y)\sigma(dy)$$

where $\sigma(dy)$ is the area measure on $\partial B(x,\delta)$. This alternative characterization of harmonic function is known as Gauss's theorem and plays a basic role in probabilistic potential theory, because probability reasoning integrates better than differentiates.

Dirichlet's problem(or first boundary value problem). Given D and a continuous function f on ∂D, to find a function φ which is continuous in $\bar{D}$ and satisfies:

$$(3) \qquad \begin{aligned} \Delta\varphi &= 0 && \text{in D,} \\ \varphi &= f && \text{on } \partial\text{D.} \end{aligned}$$

Poisson's problem. Given D and a continuous function f in D, to find a function φ which is continuous in D and satisfies

$$(4) \qquad \begin{aligned} \Delta\varphi &= f && \text{in D,} \\ \varphi &= 0 && \text{on D.} \end{aligned}$$

We have stated these problems in the original forms, of which there are well-known generalizations. As stated, a unique solution to either problem exists provided that the boundary ∂D is not too irregular. Since we shall treat only the one-dimensional case we need not be concerned with the general difficulties.

In R^1, a domain is just an bounded open nonempty interval $I = (a,b)$. Its boundary ∂I consists of the two points $\{a,b\}$. Since $\Delta f = f''$, a harmonic function is just a linear function. The boundary function f reduces to two arbitrary values assigned to the points a and b, and no question of its continuity arises. Thus in R^1 Dirichlet's problem reads as follows.

Problem 1. Given two arbitrary numbers $f(a)$ and $f(b)$, to find a function φ which is linear in (a,b) and continuous in $[a,b]$, such that $\varphi(a) = f(a)$, $\varphi(b) = f(b)$.

This is a (junior) high school problem of analytic geometry. The solution is given by

(5) $$\frac{b-x}{b-a}f(a)+\frac{x-a}{b-a}f(b).$$

Now we will write down the probabilistic solution, as follows

(6) $$\varphi(x)=E^x\{f(x(\tau))\},\quad x\in(a,b)$$

where $\tau=\tau_{(a,b)}$. If we evaluate the right member of (6) by (2.9), we see at once that it is the same as given in (5). But we will prove that φ is the sought solution by the general method developed in §3, because the same pattern of proof works in any dimension. Using the $\tau(h)$ of (3.12), we obtain

(7) $$\varphi(x)=E^x\{E^{X(\tau(h))}[f(X(\tau))]\}=\frac{1}{2}\{\varphi(x-h)+\varphi(x+h)\}$$

for any h for which (3.11) is true. This is the one-dimensional case of Gauss's criterion for harmonicity. Since φ is bounded it follows from the criterion that φ is harmonic, namely linear, But we can also involve Schwarz's Theorem in §3 to deduce this result, indeed the generalized second derivative of φ is identically zero by (7).

It remains to show that as $x\to a$ or b from inside (a,b), $\varphi(x)$ tends to $f(a)$ or $f(b)$ respectively. This is a consequence of the probabilistic relations below:

(8) $$\lim_{x\to a}P^x\{\tau=T_a\}=1,\qquad \lim_{x\to b}P^x\{\tau=T_b\}=1$$

which are immediate by (2.9). But since no such analogue is available in dimension>1, another proof more in the general spirit is indicated in Exercise 15 below. Assuming (8), we have

$$\begin{aligned}\varphi(x)&=E^x\{f(X(T_a));\tau=T_a\}+E^x\{f(X(T_b));\tau=T_b\}\\&=P^x\{\tau=T_a\}f(a)+P^x\{\tau=T_b\}f(b),\end{aligned}$$

and consequently

$$\lim_{x\to a}\varphi(x)=1\cdot f(a)+0\cdot f(b)=f(a);\quad \lim_{x\to b}\varphi(x)=0\cdot f(a)+1\cdot f(b)=f(b).$$

Thus the extension of φ to [a,b] agrees with f at a and b. [Since φ is linear in (a,b), it has a trivial continuous extension to $[a,b]$. This no longer trivial in dimension>1.]

Exercise 15. Show that for any $\varepsilon>0$:

(9) $$\lim_{x\to 0}P^x\{T_0\leqslant\varepsilon\}=1.$$

This is equivalent to $\lim_{x\to 0}P^0\{T_{-x}\leqslant\varepsilon\}=1$, and is a case of Exercise 6. Now derive (8) from (9).

Problem 2. Given a bounded continuous function f in (a,b), to find a function φ which is continuous in $[a,b]$ such that

(10) $$\frac{1}{2}\varphi''(x)=-f(x),\qquad \text{for } x\in(a,b);$$
$$\varphi(a)=\varphi(b)=0.$$

The constants $\frac{1}{2}$ and -1 in the differential equation are chosen for the sake of convenience, as will become apparent below. This is a simple calculus problem which can be solved by setting

$$\varphi(x)=\int_a^x 2(y-x)f(y)dy+cx+d$$

and determining the constants $d=-ca$ and $c=(b-a)^{-1}\int_a^b (b-y)f(y)dy$ by the boundary conditions $\varphi(a)=0$ and $\varphi(b)=0$. Substituting these values for c and d and rearranging we can write the solution above as

(11) $$\varphi(x)=\int_a^b g(x,y)f(y)dy$$

where

(12) $$g(x,y)=\begin{cases}\dfrac{2(x-a)(b-y)}{b-a}, & \text{if } a<x\leqslant y<b;\\ \dfrac{2(b-x)(y-a)}{b-a}, & \text{if } a<y\leqslant x<b.\end{cases}$$

Note that $g(x,y)>0$ in (a,b) and $g(x,y)=g(y,x)$. We put $g(x,y)=0$ outside $(a,b)\times(a,b)$. The function g is known as the Green's function for (a,b) because representing the solution of (10) in the form (11) is an example of the classical method of solving differential equations by Green's functions (see Courant and Hilbert [2; Ch. V. 14] and Exercises 17 and 18 below).

Now we will write down the probabilistic solution of Problem II, as follows:

(13) $$\varphi(x)=E^x\left\{\int_0^\tau f(x(t))dt\right\},$$

Note that the integral above may be regarded as over $(0,\tau)$ so that f need be defined in (a,b) only. Without loss of generality we may suppose $f\geqslant 0$; for the general case will follow from this case and $f=f^+-f^-$. To show that φ satisfies the differential equation, we proceed by the method of §3. We have

(14) $$\varphi(x)=E^x\left\{\left(\int_0^{\tau(h)}+\int_{\tau(h)}^{\tau}\right)f(X(t))dt\right\}$$

$$=E^x\left\{\int_0^{\tau(h)}f(X(t))dt\right\}+E^x\left\{E^{x(\tau(h))}\left[\int_0^\tau f(X(t))dt\right]\right\}.$$

Let us put

(15) $$\psi(x,h)=E^x\left\{\int_0^{\tau(h)}f(X(t))dt\right\};$$

then

(16) $$\varphi(x)=\psi(x,h)+\frac{1}{2}\{\varphi(x+h)+\varphi(x-h)\}.$$

Sinc $f\geqslant 0, \psi\geqslant 0$; also $\varphi(x)\leqslant\|f\|\ E^x\{\tau\}\leqslant\|f\|\ (b-a)^2/4$. Thus φ is continuous and concave. Now write (16) as

(17) $$\frac{\varphi(x+h)-2\varphi(x)+\varphi(x-h)}{h^2}=-\frac{2\psi(x,h)}{h^2}.$$

To calculate the limit of the right member of (17) as $h\to 0$, we note by (2.12):

(18) $$E^x\{\tau(h)\}=h^2.$$

Next we have

(19) $$\psi(x,h)-f(x)E^x\{\tau(h)\}=E^x\left\{\int_0^{\tau(h)}[f(X(t))-f(X(0))]dt\right\}.$$

Since f is continuous at x, given $\varepsilon>0$ there exists $h_0(\varepsilon)$ such that if $|y-x|\leqslant h_0(\varepsilon)$ then $|f(y)-f(x)|\leqslant\varepsilon$. Hence if $0<h<h_0$, we have $|f(X(t))-f(X(0))|\leqslant\varepsilon$ for $0\leqslant t\leqslant\tau(h)$ and so the absolute value of the right member of (19) is bounded by $E^x\{\varepsilon\tau(h)\}=\varepsilon h^2$. It follows that the left member of (19) divided by h^2 converges to zero as $h\to 0$, and consequently by (18)

(20) $$\lim_{h\to 0}\frac{\psi(x,h)}{h^2}=f(x).$$

Since φ is continuous by concavity from (16), and f is continuous by hypothesis, an application of Schwarz's Theorem yields the desired result

$$\varphi''(x)=-2f(x).$$

Furthermore since

$$|\varphi(x)|\leqslant\|f\|E^x\{\tau\},$$

$\varphi(x)$ converges to zero as $x\to a$ or $x\to b$ by (2.12). On the other hand $\varphi(a)=\varphi(b)=0$ by (2.13). Thus φ is continuous in $[a,b]$ and vanishes at the endpoints.

If we equate the two solutions of Problem II given in (12) and (13), we obtain

(21) $$E^x\left\{\int_0^{\tau}f(X(t))dt\right\}=\int_a^b g(x,y)f(y)dy$$

for every bounded continuous f on (a,b). Let us put for $x\in R^1$ and $B\in\mathscr{B}^1$:

(22) $$V(x,B)=E^x\left\{\int_0^{\tau}1_B(X(t))dt\right\}.$$

Then it follows from (21) and F. Riesz's theorem on the representation of linear functionals on (a,b) as measures (see, e. g., Royden [3, p.310]) that we have

(23) $$V(x,B)=\int_B g(x,y)dy.$$

In other words, $V(x,\cdot)$ has $g(x,\cdot)$ as its Radon-Nikodym derivative with respect to the Lebesgue measure on (a,b). The kernel V is sometimes called the potential of the Brownian motion killed at τ. It is an important object for the study of this process since $V(x,B)$ gives the expected occupation time of B starting from x.

Exercise 16. Show by using elementary calculus that the solutions to Problems I and II in R^1 are unique.

Example 17. Define a function $g(x,y)$ in $[a,b]$ as follows. For each x let $g_x(\cdot)=g(x,\cdot)$.

(i) g_x is a continuous function with $g_x(a)=g_x(b)=0$;

(ii) for all $x\neq y$, $g_x''(y)=0$;

(iii) $\lim_{\varepsilon\to 0}(g_x'(x+\varepsilon)-g_x'(x-\varepsilon))=-1$.

Show that the function $g(x,y)$ defined by (12) is the only function with these properties.

Example 18. Let the function δ_y have the defining property that for any function f on (a,b) we have

$$\int_a^b \delta_y(u)f(u)du=f(y). \tag{23}$$

This δ_y is called the **Dirac delta function** [never mind its existence!]. It follows that

$$\int_a^x \delta_y(u)du=1_{(a,x)}(y). \tag{24}$$

We can now solve the differential equation

$$h''=-2\delta_y, \qquad h(a)=h(b)=0.$$

by another integration of (24). Carry this out to obtain $h(x)=g(x,y)$, which is what results if we let $f=-2\delta_y$ in (21).

Exercise 19. Determine the measure $H(x,\cdot)$ on ∂I so that the solution to Problem 1 may be written as

$$\int_{\partial I} f(y)H(x,dy).$$

The analogue in R^d is called the harmonic measure for I. It is known in the classical theory that this measure may be obtained by taking the "interior normal derivative" of $g(x,y)$ with respect to y. Find out what this means in R^1.

Exercise 20. Give meaning to the inverse relations:

$$\frac{\Delta}{2}(-G)=\underline{I}, \qquad (-G)\frac{\Delta}{2}=\underline{I}$$

where $\underline{I}$ is the identity, and G is the operator defined by $Gf(x)=\int_a^b g(x,y)f(y)dy$.

Exercise 21. Solve the following problem which is a combination of problems 1 and 2. Given f_2 on ∂I and continuous f_1 in I, find φ such that φ is continuous in $\bar{I}$ and satisfies

$$\frac{1}{2}\varphi''=-f_1 \qquad \text{in } I,$$

$$\varphi=f_2 \qquad \text{on } \partial I.$$

§6. Feynman-Kac Functional

As a final application of the general method, we will treat a fairly new problem. Reversing the previous order of discussion, let us consider

(1) $$\varphi(x)=E^x\left\{\exp\int_0^\tau q(X(t))dt\cdot f(X(\tau))\right\}, \qquad x\in[a,b]$$

where q is a bounded continuous function in $[a,h]$, f as in Problem 1 above. Note that by Exercise 3:

(2) $$\varphi(a)=f(a),\ \ \varphi(b)=f(b).$$

The exponential factor in (1) is called the Feynman-Kac functional; see [Kac].

An immediate question is whether φ is finite. If $q\equiv$a constant c, and $f\equiv 1$, then $\varphi\equiv\infty$ for sufficiently large c, by Exercise 11.

Let us write $e(u)=\int_0^u q(X(t))dt$ for $u\geqslant 0$.

Proposition 1. Suppose $f\geqslant 0$ in (1), If $\varphi\not\equiv\infty$ in (a,b), then φ is continuous in $[a,b]$.

Proof. Let $\varphi(x_0)<\infty$, and $x\neq x_0$, $x\in(a,b)$. Then we have by the strong Markov property

$$\infty>\varphi(x_0)\geqslant E^{x_0}\{e(\tau);T_x<\tau\}=E^{x_0}\{e(T_x);T_x<\tau\}\varphi(x).$$

Since $P^{x_0}\{T_x<\tau\}>0$ and $e(T_x)>0$, this implies $\varphi(x)<\infty$.

Next, given any $A>0$, there exists $\delta>0$ such that we have

(3) $$E^x\{e^{A\tau(\delta)}\}<\infty.$$

This follows from the derivation of (1.6). Consequently we have by dominated convergence

(4) $$\lim_{h\to 0}E^x\{e^{A\tau(h)}\}=1.$$

We now state a lemma.

Lemma. Let φ be a finite nonnegative function on $[a,b]$ having the following approximate convexity property. For each $[x_1,x_2]\subset[a,b]$, $0<x_2-x_1<\delta(\varepsilon)$ and $x=\lambda x_1+(1-\lambda)x_2$, $0<\lambda<1$, then

(5) $$(1-\varepsilon)\{\lambda\varphi(x_1)+(1-\lambda)\varphi(x_2)\}<\varphi(x)<(1+\varepsilon)\{\lambda\varphi(x_1)+(1-\lambda)\varphi(x_2)\}.$$

Such a φ is continuous in $[a,b]$.

Proof of the lemma. Let $h<\delta(1)$ and $\lambda_1=\lambda_2=\frac{1}{2}$, then we have by (5)

(6) $$\varphi(x)<\{\varphi(x+h)+\varphi(x-h)\}.$$

For a suitable h we can divide $[a,b]$ into a finite number of subintervals of length 2h each. If we apply (5) to each subinterval we see that φ is bounded.

Now fix x in (a,b) and shrink $[x_1,x_2]$ to x in such a way that $\lambda_1\to 1$ and $\varphi(x_1)\to\underline{\lim}_{y\uparrow\uparrow x}\varphi(y)$ or $\varphi(x_1)\to\overline{\lim}_{y\uparrow\uparrow x}\varphi(y)$, we obtain from (5):

$$(1-\varepsilon)\ \overline{\lim_{y\uparrow\uparrow x}}\ \varphi(y)<\varphi(x)<(1+\varepsilon)\ \underline{\lim_{y\uparrow\uparrow x}}\ \varphi(y).$$

Similarly for $y\downarrow\downarrow x$. Since ε is arbitrary this shows that φ is continuous at x. A similar argument shows that φ is unilaterally continuous at a and at b. Lemma is proved.

We return to φ and generalize the basic argument in §3 by considering the first exit time from an asymmetric interval $(x-h, x+h')\subset(a,b)$, starting from x. Recall that

(7) $$P^x\{T_{x-h}<T_{x+h'}\}=\frac{h'}{h+h'}.$$

For sufficiently small h and h′ we have by (4):

(8) $$1-\varepsilon<E^x\{e^{-Q\tau^*}\},\quad E^x\{e^{Q\tau^*}\}<1+\varepsilon$$

where $Q=\|q\|$ and $\tau^*=\tau_{(x-h\quad x+h')}$. The strong Markov property yields, for the φ in (1) with an arbitrary f:

(9) $$\varphi(x)=E^x\{e(\tau^*)\}\{\frac{h'}{h+h'}\varphi(x-h)+\frac{h}{h+h'}\varphi(x+h')\}.$$

Using (8) and (9) we see that if $f\geqslant 0$ in (1) then φ satisfies the conditions of the Lemma, and is therefore continuous in $[a,b]$. In particular this is true when $f=1_{\{a\}}$ or $1_{\{b\}}$. Hence it is also true for the φ in (1) for an arbitrary finite f. Proposition 1 is proved.

Let us write φ_a and φ_b for the φ in (1) when $f=1_{\{a\}}$ and $f=1_{\{b\}}$ respectively. According to Proposition 1, either $\varphi_a\equiv\infty$ or φ_a is bounded continuous in $[a,b]$, and similarly for φ_b. However, it seems possible that $\varphi_a\equiv\infty$ but $\varphi_b\not\equiv\infty$ in $[a,b]$, or vice versa. For a general f, we have

(10) $$\varphi(x)=f(a)\varphi_a(x)+f(b)\varphi_b(x),$$

provided the right member above is not $+\infty-\infty$ or $-\infty+\infty$. This is certainly the case under the hypothesis of the next proposition.

Proposition 2. Suppose that $\varphi_a\not\equiv\infty$ and $\varphi_b\not\equiv\infty$ in (a,b). Then for any $f\geqslant 0$ we have

$$\frac{1}{2}\varphi''+q\varphi=0$$

in (a,b), and φ is continuous in $[a,b]$.

Proof, Write

(11) $$E^x\{e(\tau(h))\}=1+\psi(x,h);$$

then equation (9) for $h=h'$ takes the form:

(12) $$\frac{\varphi(x+h)-2\varphi(x)+\varphi(x-h)}{h^2}=\frac{\psi(x,h)}{h^2}\{\varphi(x+h)+\varphi(x-h)\}.$$

Since we have proved that φ is continuous, the quantity in (12) will converge to $-\lim_{h\to 0}[\psi(x,h)/h^2]\, 2\varphi(x)$ as $h\to 0$, provided that the latter limit exists. To show this we need

(13) $$E^x\{\tau(h)^2\} = \frac{5}{3}h^4;$$

also that for sufficiently small h we have by (4):

(14) $$E^x\{e^{4(Q+1)\tau(h)}\} \leqslant 2.$$

Exercise 22. Prove (13). Can you get a general formula for $E^x\{\tau(h)^k\}, k\geqslant 1$?

Using the trivial inequality $\sqrt{u}\leqslant e^u$ for all $0\leqslant u<\infty$, we have

$$\tau(h)^2 e^{Q\tau(h)} \leqslant \tau(h)^{3/2} e^{(Q+1)\tau(h)}.$$

Hence by Hölder's inequality (13) and (14),

(15) $$E^x\{\tau(h)^2 e^{Q\tau(h)}\} \leqslant E^x\{\tau(h)^2\}^{3/4} E^x\{e^{4(Q+1)\tau(h)}\}^{1/4} \leqslant c_1 h^3,$$

where c_1 is a constant. Next we use the inequality

$$|e^u - 1 - u| \leqslant \frac{u^2}{2} e^{|u|},$$

valid for all u, to obtain

$$E^x\{\,|e(\tau(h)) - 1 - \int_0^{\tau(h)} q(X(t))dt|\,\}$$

$$\leqslant \frac{1}{2}E^x\{(\int_0^{\tau(h)} q(X(t))dt)^2 e^{Q\tau(h)}\} \leqslant \frac{Q^2}{2} E^x\{\tau(h)^2 e^{Q\tau(h)}\}.$$

The last term divided by h^2 converges to zero as $h\to 0$, by (15). Hence by (20) of §5 with f replaced by q:

$$\lim_{h\to 0}\frac{\psi(x,h)}{h^2} = \lim_{h\to 0}\frac{1}{h^2}E^x\left\{\int_0^{\tau(h)} q(X(t))dt\right\} = q(x).$$

Therefore Schwarz's theorem applied to (12) yields $\varphi'' = -2q$ as asserted. Note that the continuity of φ is required here also. Proposition 3 is proved.

Propositions 1 and 2 together give a dichotomic criterion for the solvability of the following problem.

Problem 3. Given a bounded continuous function q in (a,b) and two arbitrary numbers $f(a)$ and $f(b)$, to find a function φ which is continuous in $[a,b]$ such that

(16) $$\frac{1}{2}\varphi''(x) + q(x)\varphi(x) = 0, \qquad x\in(a,b);$$

$$\varphi(a) = f(a), \qquad \varphi(b) = f(b).$$

Exercise 23. Is the solution to Problem III unique when it exists?

Exercise 24. Solve the problem similar to Problem 3 but with the right side of the differential equation in (16) replaced by a given bounded continuous function in (a,b). This is the Poisson problem with the Feynman-Kac functional.

Exercise 25. Prove that if the equation in (16) has a positive solution φ in (a,b), then for any $[c,d]\subset(a,b)$, we have

$$\varphi(x)=E^x\{e(\tau_{(c,d)})\varphi(X(\tau_{(c,d)}))\},\qquad x\in[c,d].$$

In particular,

$$x\to E^x\{e(\tau_{(c,d)})\}$$

is bounded in $[c,d]$.

Exercise 26. Prove that if the differential equation in (16) has a positive solution in each interval (c,d) such that $[c,d]\subset(a,b)$ (without any condition on the boundary $\{c,d\}$) then it has a positive solution in (a,b). These solutions are a prior unrelated to one another.

Exercise 27. Is it possible that $\varphi_a\equiv\infty$ in (a,b) whereas $\varphi_b\not\equiv\infty$ in (a,b)? Here φ_a and φ_b are defined before Proposition 3. This is a very interesting problem solved by M. Hogans a graduate student at Stanford.

References

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