1982年 第2卷 第4期
Vol.2, No.4, 1982

数学研究与评论
JOURNAL OF MATHEMATICAL RESEARCH & EXPOSITION

A SURVEY OF SOME ASPECTS ON THE RESEARCH WORK OF FUZZY TOPOLOGY IN CHINA*

Pu Bao-ming (蒲保明) and Liu Ying-ming (刘应明)

(Institute of Mathematics, Sichuan University)

The fundamental concept of a fuzzy set, introduced by Zadeh in 1965 [34] provides a natural foundation for treating mathematically the fuzzy phenomena which exist prevasively in our real world and for building new branches of fuzzy mathematics. In the area of fuzzy topology much research has been carried out since 1968 [1].

Early in the 1970's the authors noticed the study of fuzzy topology and were interested in it. Later on, members of the research group of topology in Sichuan University launched investigation on fuzzy topology under the support and encouragement of Professor Kwan Chao-Chih, the director of Institute of Systems Science and Mathematical Sciences of the Academia Sinica. Today, much research work on fuzzy topology are carried on in many parts of China.

In paper [25], two fundamental problems of fuzzy topology were solved: one problem concerns the proper definition of fuzzy points and their neighborhood structure and the other problem is the establishment of Moore-Smith convergence theory in the fuzzy topological spaces. The authors discovered that for the neighborhood structure of fuzzy points, in addition to the traditional concept of neighborhoods and the usual belonging relation, $\in$, it was needed to introduce a new kind of

* Received Dec. 12, 1981.

neighborhood structure, called Q—neighborhoods, for fuzzy points, and a new kind of relation, called Q—relation (the quasi-coincident relation) between fuzzy points and fuzzy sets, In an ordinary topological space, as a special case of a fuzzy topological space, these concepts: neighborhood system and Q—neighborhood system, $\in$—relation and Q—relation coincide respectively. With these new concepts as main tools, they generalized all the theorems in the chapters I and II of the celebrated book on general topology [12] to fuzzy topological spaces with exception at most two less important ones. This means that these two problems in fuzzy topology have been solved in almost the same degree as the corresponding problems in General Topology had been solved.

In [22], some mutually equivalent systems of axioms, which seem intuitively to be evident, for establishing the neighborhood structure were given and the theorem that in a L—fuzzy topological space the unique neighborhood structure satisfying any one of these systems of axioms is exactly the Q—neighborhood system given in [25] was proved. This fact thus throws light on the limitation of the traditional neighborhood system and the reasonableness and the intrinsic property of the Q—neighborhood system. Considering those L—fuzzy topological spaces where L is a lattice with some special properties, the paper [15] says that in this class of L—fuzzy topological spaces, there is a certain complementary property between the neighborhood system and the Q—neighborhood system.

In [30, 31], Wang introduce two nice concepts: "fuzzy topological molecular lattice" and "far neighborhood" (which is a complement of Q—neighborhood in some sense) from a more abstract point of view and obtained many interesting results therein. Using Q—neighborhood as the basic tool, the theory of Moore-Smith convergence was established in [25]. But [30] generalized this theory into the topological molecular lattices.

In [14], a new proof of the C. T. Yang's theorem concerning derived sets in the fuzzy topological spaces was given. This proof is much simpler than those given in [25] and [30]. In [29], a revised definition of fuzzy boundary was given in terms of the concept of the Q—neighborhood and was used to study the dimension theory of fuzzy topological spaces [28]. The paper [9] introduced a new definition of fuzzy subspaces. It is more general than that given in [25].

As for the fuzzy product spaces and quotient spaces, [26] completed the generalizations of all the theorems in Chapter Ⅲ of the famous book [12].

Let us now turn to the aspect of fuzzy compactness. In the literature a lot of different kinds of fuzzy compactness notions have been introduced and studied.(e.g. see [13]) In [7], fuzzy compactness [1] was first characterized in term of the Moore-Smith convergence of fuzzy nets. In [16] a mistake in [3] was pointed out and a counterexample was constructed. An open problem concerning α^*—compactness posed there had been fundamentally solved also. So far as we know, the interesting papers on fuzzy compactness notions may be [17] and [32]. In [32], Wang first defined the notion of α—nets and then introduced one new kind of fuzzy compactness, called nice fuzzy compactness (or simply, nice compactness), by using α—net from the point of view of convergence. Nice fuzzy compactness has almost all the properties which the ordinary compactness has in general topology. Moreover, this paper gave a clear analysis and a dear description of all the relations and differences among a variety of fuzzy compactness notions given in the foreign literature, inclusive of nice fuzzy compactness. From the standpoint of the net convergence, the definition of nice fuzzy compactness may be considered to be given by analyzing all the α—levels simultaneously. However, the other new fuzzy compactness notion given in [17] was defined in term of the open Q—covers, based upon Q—neighborhoods. This compactness, called Q—compactness, is considered to have less requirements in the sense that it can be characterized in term of the convergence property of "maximum fuzzy point" of the fuzzy set only. Among the nice properties Q—compactness enjoys, the Tychonoff product theorem for Q—compactness is fundamental one. Within the framework of the better fuzzy compactness notions, i. e. nice compactness and Q—compactness, the fundamental results concerning compactness of Chapter 5 of [12] have been generalized to fuzzy topological spaces.

As to the compactification of fuzzy topological spaces, there were already some works [24, 3] abroad, but the general theory of fuzzy Stone-Čech compactification was recently finished in [21] by using the fuzzy embedding theorem [20] and the nice compactness [30]. Adding a weaker separation axiom, called sub—T_0 axiom, to a fuzzy completely regular space [6], the concept of fuzzy Tychonoff space is defined [20]. In [21] it is proved that for every fuzzy Tychonoff space, there exists a fuzzy compactification, called fuzzy Stone-Čech compactification, which has the continuous extension property of continuous mappings similar to that of the usual Stone-Čech compactification in general topology.

In the area of the fuzzy uniformity, the New-Zealand Mathematian Hutton has done a piece of profound and penetrating work [6]. However, the author of [18] has pointed out that the fundamental formula, occuring in Lemma 3 of [6] and often used in this area, does not hold for almost all the completely distributive lattices and proved that it holds good under a simple natural additional condition. From the algebraic point of view, the paper [18] is an investigation concerning the intersection operation on union-preserving mappings in completely distributive lattices. In [19], a further discussion on the inverse operation on the above mentioned mappings was made. These works just mentioned above provide useful algebraic tool for investigation of fuzzy uniformities. The author of [20] contributed three things to fuzzy topology. Firstly, he gave a sound proof of the fuzzy Weil theorem (i. e. a fuzzy topological space is fuzzy uniformizable iff it is fuzzy completely regular.) This theorem was first given in [6], but there are some drawbacks in the original proof. Secondly, he gave a pointwise characterization of fuzzy completely regularity by means of the concept Q—neighborhoods. This characterization thus provides a useful tool for establishing the fuzzy embedding theorem. (Hence it may be seen that in the fuzzy imbedding theory, the concept of fuzzy points cannot be avoided and therefore the point of view of the "point-free" treatment of fuzzy topology has its own limitation.) Finally he proved the general fuzzy embedding theorem: a L—fuzzy topological space is a Tychonoff space iff it can be embedded in the product of family of fuzzy interval, i. e. a fuzzy basic cube.

A general theory of fuzzy metric spaces in [2] is considered to be successful in a certain sense. In [5] a kind of metric for the fuzzy topological spaces was obtained. The fuzzy p. q. metric space has been introduced by Hutton [6] and Erceg [2] respectively. In [37], a connection between these two definition of fuzzy p. q. metric spaces was completely described. Also, the product space of fuzzy p. q. metric spaces and $Q—C_I$ of p. q. metric space were investigated. Finally, the fuzzy Urysohn metrizable theorem has succesfully established.

In [36], a theory of fuzzy function spaces has been established. In the space, the fuzzy pointwise topology and fuzzy compact open topology were introduced. The joint continuity and some separation properties of the space, such as the completely regularity, were investigated.

In [11], seme basic cardinal functions such as weight, charactor, density and cellularity were introduced in fuzzy topological spaces. Among others, the famous Hewitt-Marczewskipondiczery theorem concerning the density of cartisian products was generalized to fuzzy topological spaces.

As is well-known, fuzzy topology is built up on fuzzy set theory, as its foundation. Now fuzzy topology has developed in such a extent that it can react upon its fundation. That is to say, the results obtained thus far in fuzzy topology have important applications in some other branches of fuzzy set theory. Take, for instance, the theory of convex fuzzy sets. In the basic and classical paper [34], Zadeh used almost second half of it to discuss the fuzzy convex sets. There are two main results in this aspect which are worth while to be mentioned. The first one is concerning a property of the shadow of fuzzy convex sets and the other is the separation theorem for fuzzy convex sets. The author of [33] gave a counter-example showing that Zadeh's second result mentioned above cannot be correct even for ordinary (crisp) convex sets and recasted this result by emloying the induced fuzzy topology. Later on, the author of [23] gave another counterexample showing that Zadeh's first result is also not true and recasted it by adding some topological conditions to give several positive results.

Finally, we would like to mention briefly two other aspects of research on fuzzy topology in China. Much work on induced fuzzy topology in the sense of Weiss [33] has been done [4, 8, 27, 35]. The dimension theory for fuzzy topology has been initiated [28].

References

[1] C. L. Chang, Fuzzy topological spaces, *J. Math. Anal. Appl.*, 24 (1968), 191—201.

[2] M. A. Erceg, Metric spaces in fuzzy set theory, J. Math. *Anal. Appl.*, 69 (1979), 205—230.

[3] T. E. Gantner and R. C. Steinlage and R. H. Warren, Compactness in fuzzy topological spaces, *J. Math. Anal. Appl.*, 62 (1978), 547—562.

[4] Hu Cheng-Ming, A class of fuzzy topological spaces I, II, *Neimenggu Daxue Xuebao* 1(1981), (in Chinese).

[5] ________________, A metrization of fuzzy topological spaces, *Ziran Zazhi* 7 (1981), 554. (in Chinese)

[6] B. Hutton, Uniformities on fuzzy topological spaces, *J. Math. Anal Appl.*, 58 (1977), 559—571.

[7] Jiang Ji-Quang, Separation axioms in fuzzy topological spaces and fuzzy compactness, *Sichuan Daxue Xuebao* 3 (1979), 1—10. (in Chinese).

[8] ________________, Concerning separation axioms for induced fuzzy topological spaces, *Trans. Sichuan Math. Soc.*, 1 (1981), 25—31. (in Chinese).

[9] ________________, On fuzzy countably compact spaces and fuzzy sequentially compact spaces, *Sichuan Daxue Xuebao*, 1 (1980), 37—43. (in Chinese).

[10] ________________, Fuzzy paracompact spaces, *ibid*, 3 (1981), 47—52. (in Chinese).

[11] ________________, Weight, charactor, and density of L—fuzzy topological spaces, *Acta Mathematica Scientia*, (to appear) (in Chinese).

[12] J. L. Kelley, General Topology. (Van Nostrand, 1955).

[13] R. Lowen, A comparison of different compactness notions in fuzzy topological spaces, *ibid.*, 64 (1978), 446—454.

[14] Li Zhongfu, On the generalization of C. T. Yang's theorem concerning derived set in fuzzy topology, *Fuzzy Math.*, 1 (1981), 39—42. (in Chinese).

[15] ________________, A class of L—fuzzy topological spaces I. Neighborhood structures of L—fuzzy points, *J. Math. Research and Exposition*, 2(1982), 1. 37—44. (in Chinese).

[16] Liu Ying-ming, A note on compactness in fuzzy unit interval. *Kexux Tongbao* 25 (1980). A

special issue of Math., Phy.. and Chem.. 33—35. (in Chinese).

[17] Liu Ying-ming, Compactness and Tychonoff theorem in fuzzy topological spaces. *Acta Mathematica Sinica*, 24(1981), 260—268. (in Chinese).

[18] ________, Intersection operation on union-perserving mappings in completely distributive lattices, *J. Math. Anal. Appl.*, 84 (1981), 249—255.

[19] ________, Inverse operation on union-preserving mappings in lattices and its applications to fuzzy uniform spaces, (to appear in Proc. of 12th International Symposium on Multipl-Valued Logic).

[20] ________, A pointwise characterization of fuzzy completely regularity and imbedding theorem in fuzzy topological spaces, *Scientia Sinica* (to appear).

[21] ________, On the fuzzy Stone-Cech compactification, Acta Math. Sinica (to appear) (in Chinese), A abstract see *Kexue Tongbao*, 27 (1982), 2: 127. (in Chinese).

[22] ________, Neighborhood structures in fuzzy topological spaces, *Kexue Tongbao*, 27 (1982), 3: 189—190 (in Chinese).

[23] ________, some properties of fuzzy convex sets, (This Volume) (A detailed abstract in Chinese has been published in *Kexue Tongbao* 27 (1982), 6: 328—330.

[24] H. W. Martin, A Stone-Cech ultra fuzzy compactification, *J. Math. Anal. Appl.*, 73 (1980), 453—456.

[25] Pu Pao-Ming and Liu Ying-Ming, Fuzzy topology I, Neighborhood structure of a fuzzy point and Moore-Smith convergence, *J. Math. Anal. Appl.*, 76 (1980), 571—599. (*Sichuan Daxue* xuebao 1 (1977), 31—50. (in Chinese)).

[26] Pu Pao-Ming and Liu Ying-Ming, Fuzzy topology II. product and quotient spaces, J. Math. Anal. Appl., 77 (1980), 20—37. (A detailed abstract in Chinese has been published in *Kexue Tongbao* 24 (1979), No. 3, 97—100.

[27] Pu Si-Li, Mapping properties of induced fuzzy topological spaces (to appear) (in Chinese).

[28] ________, Zero-dimensional fuzzy topological spaces (to appear) (in Chinese).

[29] ________, A better Definition on fuzzy boundary, Fuzzy Math., *Fuzzy Mathematics*, 2 (1982) No. 1 (in Chinese).

[30] Wang Guo-Jun, Topological molecular lattices (I), *Sanshi Sida Xuebao*, 1 (1979) 1—15. (in Chinese).

[31] ________, Separation axioms in topological molecular lattices, *J. Math. Research and Exposition*, (to appear)

[32] ________, A new fuzzy compactness defined by fuzzy Nets. *J. Math. Anal. Appl.*, (to appear)

[33] M. D. Weiss, Fixed points, separation and induced topologies for fuzzy sets, *J. Math. Anal. Appl.*, 50 (1975), 142—150.

[34] L. A. Zadeh, Fuzzy sets, *Inform. control*, 8 (1965), 338—353.

[35] Zhou Hao-Xuan, Relations between topological spaces and fuzzy topological spaces, (in manuscript).

[36] Pung Yuwei, Fuzzy function spaces, (to appear), (in Chinese)

[37] Liang Jihua, Some problems on fuzzy metric spaces, (to appear), (in Chinese).