第3卷 第1期
1983年 1月

数 学 研 究 与 评 论
JOURNAL OF MATHEMATICAL
RESEARCH AND EXPOSITION

Vol.3 No.1
Jan. 1983

On the Diophantine Equation $x^4-Dy^2=1$*

C. D. Kang (康继鼎) *D. Q. Wan* (万大庆) *G. F. Chou* (周国富)

(Chengdu Cillege of Geology)

The research of the Diophantine equation

$$x^4-Dy^2=1 \tag{1}$$

was started by Ljunggren in 1942, where $D>0$ and is not a square integer.

Since then thanks to the work of Ljunggren, Cohn, Bumby, and Ko Chao(柯召). Sun Chi (孙琦), many advances of the research have been made. One can refer to [1] and its references.

In this paper, the main theorems in [1] are improved by us, that is, we have the following

Theorem A Let $D=P_1\cdots P_s\not\equiv 7 \pmod 8$, $P_1\equiv 1 \pmod 4$, $P_i\equiv 3 \pmod 4$ $(2\leqslant i\leqslant s)$, where $P_1,\cdots,P_s$ are distinct odd primes. Then, when

(i) $2P_1=a^2+b^2$, $a\equiv\pm 3 \pmod 8$, $b\equiv\pm 3 \pmod 8$, or

(ii) $\left(\frac{P_j}{P_1}\right)=-1$ for some $j(2\leqslant j\leqslant s)$,

Eq. (1) has no positive integer solutions.

Theorem B Let $D=2P_1\cdots P_s$, $P_1\equiv 1 \pmod 4$, $P_i\equiv 3 \pmod 4$ $(2\leqslant i\leqslant s)$, where $P_1,\cdots,P_s$ are distinct odd primes. Then, when

(i) $2P_1=a^2+b^2$, $a\equiv\pm 3 \pmod 8$, $b\equiv\pm 3 \pmod 8$, or

(ii) $\left(\frac{P_j}{P_1}\right)=-1$ for some $j(2\leqslant j\leqslant s)$, or

(iii) $P_1\equiv 5 \pmod 8$

Eq. (1) has no positive integer solutions except the solutions $x=47321$, $y=5219916$ $(D=2\cdot 5\cdot 7\cdot 11\cdot 239)$ and $x=41$, $y=116$ $(D=2\cdot 3\cdot 5\cdot 7)$.

To prove the theorems, we need the following Lemma which can be derived by induction on the size of h and using some well-known results in the theory of Diophantine equations.

* Received Sept. 15, 1981.

Lemma Let $h=h_1h_2$, greater than one, be a positive integer and both its two prime factors be of the form $4k+3$. Then the equation

$$h_1^2x^4-8h_2^2y^4=1 \tag{2}$$

has no positive integer solutions except the solutions $x=y=1$, $h=h_1=3$, $h_2=1$; and the equation

$$x^2-8h^2y^4=1, \quad h\neq 239 \tag{3}$$

has no positive integer solutions.

Now we begin to prove our theorems by providing the outlines.

Proof of theorem A As [1], from (1) we can obtain

$$x^2+1=2u^2,\ x^2-1=2D(2v)^2,\ y=4uv,\ (u,\ v)=1. \tag{4}$$

Further we have

$$u+1=2lt^2,\ u-1=2ks^2,\ D=lk,\ v=st,\ (l,\ k)=(s,\ t)=1. \tag{5}$$

When P_1 is a factor of k, we can give a contradiction by using the methods in [1]. when P_1 is not a factor of k, from(4)、(5), we have

$$e_1^2m_1^4+1=2(l_1m_1^2-f_1n_1^2)^2,\ k=e_1f_1,\ s=m_1n_1,\ (e_1,\ f_1)=(m_1,\ n_1)=1. \tag{6}$$

By the theory of Pell equations, from(6), we have

$$f_1n_1^2=e_1m_1^2-(e_1m_1^2-f_1n_1^2)=2\cdot\frac{\varepsilon^b+\bar{\varepsilon}^b}{\varepsilon+\bar{\varepsilon}}\cdot\frac{\varepsilon^b-\bar{\varepsilon}^b}{\varepsilon-\bar{\varepsilon}}, \tag{7}$$

where b is a positive integer, $\varepsilon=1+\sqrt{2}$.

We may change (7) into the form of Eq. (2) and so far the proof is clearly completed.

The proof of theorem B is similar to that of theorem A and is omitted.

Reference

[1] Ko Chao and Sun Chi, On the Diophantine equation $x^4-Dy^2=1$, Chinese Annals of Mathematics, 1 (1980). 83—89.