第3卷 第1期
1983年 1月

数 学 研 究 与 评 论
JOURNAL OF MATHEMATICAL
RESEARCH AND EXPOSITION

Vol. 3 No. 1
Jan. 1983

On the Maximality of Certain Orthogonal Groups Embedded in Symplectic and Unitary Groups resp.*

Li Shang-zhi (李尚志)

(China University of Science and Technology)

It is of interest to study the maximality of a classical group embedded in another. We have proved in [1] the maximality of special generalized symplectic (unitary, orthogonal resp.) groups in corresponding special linear groups. And this paper is a sketch of a section of [1] devoted to another two cases.

I. The maximality of $O(2m,2^k,Q)$ in $\mathrm{SP}(2m,2^k)$

Here the symplectic inner product $(x,y)=Q(x+y)+Q(x)+Q(y)$ in the underlying space V.

Theorem 1 $G_0=O(2m,2^k,Q)$ is maximal in $G=\mathrm{SP}(2m,2^k)$.

Proof Each symplectic transvection in G can be written in the form $t_v: z\longmapsto z+(z,v)v$, $z\in V$, with the uniquely determined vector v. $t_v\in G_0$ iff $Q(v)=1$, and any subgroup $X\supsetneqq G_0$ of G contains some t_{v_1} with $Q(v_1)=s\neq 1$. write $v_1=se+f$ with $Q(e)=Q(f)=0$, $(e,f)=1$ and take $x_1=ce+x_2$, $c\in F_{2^k}$, $x_2\in\langle e,f\rangle^\perp$ and $Q(x_2)=1$, then X contains $t_{v_1}^{-1}t_{x_1}t_{v_1}=t_x$, where $x=x_1t_{v_1}=x_1+(x_1,v_1)v_1$, $Q(x)=1+c^2(s+1)$ runs over F_{2^k} as c does, which implies that X contains all symplectic transvections and hence is equal to G.

II. The maximality of an orthogonal group contained in $\mathrm{PSU}(n,q^2)$ (q odd, $n\geqslant 3$)

In [2], we proved the maximality of the nomalizer of $\mathrm{PSp}(2m,q)=\mathrm{PSL}(2m,q)\cap\mathrm{PSU}(2m,q^2,\begin{pmatrix} & I_m\\ -I_m & \end{pmatrix})$ in $\mathrm{PSU}(2m,q^2,\begin{pmatrix} & I_m\\ -I_m & \end{pmatrix})$.

*Received June 4, 1981.

Let
$$H=\begin{cases}\begin{pmatrix} & I_m\\ I_m & \end{pmatrix}, & \text{when } n=2m\\ \begin{pmatrix} & I_m & \\ I_m & & \\ & & 1\end{pmatrix}, & \text{when } n=2m+1.\end{cases}$$

We have now $\mathrm{PSL}(n,q)\cap\mathrm{PSU}(n,q^2,H)=\mathrm{PSO}(n,q,H)$ and

Theorem 2 Let $G=\mathrm{PSU}(n,q^2,H)$ $(n\geqslant 3)$, $G\neq\mathrm{PSU}(3,9)$, $\mathrm{PSU}(3,25)$, $\mathrm{PSU}(4,9)$, then the normalizer $G_{\langle q\rangle}=G\cap\mathrm{PGO}(n,q,H)$ of $G_q=\mathrm{PSO}(n,q,H)$ in G is maximal in G.

Consider the subset $V_q=\{(a_1,\cdots,a_n)\,|\,a_i\in F_q,\ (1\leqslant i\leqslant n)\}$ of the underlying space $V=\{(a_1,\cdots,a_n)\,|\,a_i\in F_{q^2},\ (1\leqslant i\leqslant n)\}$ and the set of lines $V_{\langle q\rangle}=\{\langle x\rangle=F_{q^2}x\,|\,x\in V_q\}$, then G_q and $G_{\langle q\rangle}$ are the stabilizers of V_q and $V_{\langle q\rangle}$ in G resp.. Let $\Delta\in F^*_{q^2}$, $\bar\Delta=-\Delta$. As in [2], for any isotropic vector u in V, the subgroup $T_u=\{t_{u,s}:z\longmapsto z+s\Delta(z,u)u,\ z\in V\,|\,s\in F_q\}$ consisting of unitary transvections is called the T-subgroup corresponding to ⟨u⟩. And furthermore, when $n\geqslant 4$, we can take an orthogonal pair of non-collinear isotropic vectors u, x in V and define a subgroup

$$T_{\langle u,x\rangle_q}=\{t_{u,sx}:z\to z+(z,sx)u-(z,u)sx,\ z\in V\,|\,s\in F_q\}$$

called the $T^{\langle 2\rangle}$-subgroup corresponding to the F_q-plane $\langle u,x\rangle_q=F_qu+F_qx$. We note that $T_{\langle u,x\rangle_q}<G_{\langle q\rangle}$ iff $u=cu_1$, $x=cx_1$ for some $c\in F^*_{q^2}$, u_1, $x_1\in V_q$.

Lemma i) $\langle T_{\langle u,x\rangle_q},\ T_{\langle u,x+cu\rangle_q}\rangle>T_u$, if $c\in F_{q^2}\backslash F_q$.

ii) $\langle T_{\langle u,x\rangle_q},\ T_{\langle u,y\rangle_q}\rangle>T_u$, if $(x,y)\notin F_q$.

Proof i) $t^{-1}_{u,sx}t_{u,s(x+cu)}=t_{u,s(\bar c-c)}$ for any $s\in F_q$.

ii) $t^{-1}_{u,y}T_{\langle u,x\rangle_q}t_{u,y}=T_{\langle u,x+cu\rangle_q}$, where $c=(x,y)\notin F_q$, and then apply i).

Proof of Theorem 2 Any subgroup $X\gneqq G_{\langle q\rangle}$ of G is to be proved to contain some hence all T-subgroups T_u, $u\in V_q$, which will lead to $X=G$ by Lemmas 3 and 4 of [2].

Case I $n\geqslant 4$. Any $g\in X\backslash G_{\langle q\rangle}$ sends some isotropic line in $G_{\langle q\rangle}$ out. If we can choose g such that $G_{\langle q\rangle}g\cap G_{\langle q\rangle}$ contains isotropic lines, we can find an orthogonal pair of isotropic vectors u_0, u in V_q such that $u_0g\in V_{\langle q\rangle}$, $ug\notin V_{\langle q\rangle}$ and can write $u_0g=cx_0$, $ug=c(x+\Delta y)$ with $c\in F^*_{q^2}$, $0\neq x_0,x,y\in V_q$. $0=(x+\Delta y,x_0)=(x,x_0)+\Delta(y,x_0)$ implies $(x,x_0)=(y,x_0)=0$. X contains $g^{-1}T_{\langle u_0,u\rangle_q}g=T_{\langle x_0,x+\Delta y\rangle_q}$. When $\langle y\rangle=\langle x_0\rangle$ we have $T_{x_0}<X$ by lemma i). Otherwise, there exists an isotropic $y_0\in(x_0^\perp\cap V_q)\backslash y^\perp$ hence $(x+\Delta y,y_0)\notin F_q$ leads to $T_{x_0}<X$ by lemma ii). Now suppose $G_{\langle q\rangle}g\cap G_{\langle q\rangle}$ contains no isotropic line for any $g\in X\backslash G_{\langle q\rangle}$. X contains the conjugate $T_{\langle z_1,z_2\rangle_q}$ of $T_{\langle u_1,u_2\rangle_q}<G_q$, where $z_1=u_1g=x_1+\Delta y_1$, $z_2=u_2g=x_2+\Delta y_2$ $0\neq u_i,x_i,\ y_i\in V_q$, $i=1,2$. $\langle x_1,x_2,y_1,y_2\rangle$ cannot be totally isotropic (otherwise, we have isotropic line

$\langle x_1\rangle \in V_{\langle q\rangle}\cap V_{\langle q\rangle}t_{z_1,z_2}$ but $t_{z_1,z_2}\in X\backslash G_{\langle q\rangle}$), and we may suppose the plane $\langle x_1,y_1\rangle$ to be non-singular, $z_2=az_1+\tilde{z}_2$ with isotropic component $\tilde{z}_2=\tilde{x}_2+\Delta\tilde{y}_2$ in $\langle x_1,y_1\rangle^{\perp}$ and $\tilde{x}_2,\tilde{y}_2\in\langle x_1,y_1\rangle^{\perp}\cap V_q$ The plane $\langle\tilde{x}_2,\tilde{y}_2\rangle$ should also be non-singular and has a base $\{h_1,h_2\}\subset V_q$ with $h_1\perp h_2$, $(h_i,h_i)=c_i\neq 0$ $(i=1,2)$. Let S_x denote the symmetry $z\longmapsto z-\frac{2(z,x)}{(x,x)}x$ $(z\in V)$ determined by anisotropic vector x. When $q>3$, take an anisotropic h_1+sh_2 $(s\in F_q^*)$ and put $g_1=S_{h_1}S_{h_1+sh_2}$; when $q=3$ but $n>4$, take an anisotropic $h_1+h_2+h_3$ $(h_3\in\langle x_1,y_1,\tilde{x}_2,\tilde{y}_2\rangle^{\perp}\cap V_q)$ and put $g_1=S_{h_1}S_{h_1+h_2+h_3}$, then we have $(z_2,z_2g_1)\notin F_q$ which leads to $X>T_{z_1}$, hence $X>T_{z_1g^{-1}}=T_{u_1}$.

Case II $n=3$. $-S_x\in G_q$ iff $\langle x\rangle\in V_{\langle q\rangle}$. X contains some $-S_x$ with $\langle x\rangle=\langle(a_1,a_2,a_3)\rangle\notin V_q$. When $a_1a_2=0$, say $x=\left(0,\frac{c}{2},1\right)$, $c\notin F_q$. The commutators of $S_{(0,0,1)}S_{(0,c/2,1)}$ and $S_{(0,0,1)}S_{(0,s/2,1)}$ $(s\in F_q)$ which are in X form the T-subgroup $T_{(0,1,0)}$. In other cases, say $a_1=1$, by acting on $\langle x\rangle$ by some $\begin{pmatrix}1 & & \\ -\frac{s_1^2}{2} & 1 & -s_1\\ s_1 & & 1\end{pmatrix}$, $\begin{pmatrix}1 & -\frac{s_2^2}{2} & s_2\\ & 1 & \\ & -s_2 & 1\end{pmatrix}$ $(s_1,s_2\in F_q)$ in G_q we may suppose $a_3\notin F_q$, $a_2\in F_q$. Replace X by a suitable $\left(1,\frac{sa_3\bar{a}_3}{2},0\right)(-S_x)$ $(s\in F_q)$, we can always come back to a treated case.

References

[1] Li Shangzhi, On the subgroup systems of certain finite simple groups, Ph. D. Thesis, China University of Science and Technology (1981),

[2] Li Shangzhi & Zha Jianguo, On certain classes of maximal subgroups in PSU(n,q^2). Scientia Sinica, 1982, 2: 125-131.