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On the Approximation of Continuous Function by Harmonic Means*

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Let $f(x)$ be a continuous and periodic function with period 2π and

$$\mathfrak{S}[f]=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

be its Fourier series. Denote by $s_n(f,x)$ the n-th partial sums of $\mathfrak{S}[f]$ and by $\omega(f,\delta)$ the moduls of continuity of $f(x)$. When $\omega(\delta)$ is a modul of continuity, we denote by $H[\omega]_c$ the class of all functions for which $\omega(f,t)\leqslant\omega(t)$.

It is well known that the n-th harmonic means and the n-th Cesáro means of $f\in C_{2\pi}$ are defined as

$$N_n(f,x)=\frac{1}{P_n}\sum_{\nu=0}^{n}\frac{s_\nu(f,x)}{n-\nu+1}\qquad\left(P_n=1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$$

and

$$\sigma_n^\alpha(f,x)=\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n}A_{n-\nu}^{\alpha-1}S_\nu(f,x)\qquad\left(\alpha>-1,A_n^\alpha=\frac{(\alpha+1)\cdots(\alpha+n)}{n!}\right)$$

respectively.

A. B. Effimov[2] considered the approximation of continuous function by its partial sums and proved the following

Theorem A Let $f(x)\in H[\omega]_c$. Then

$$\sup_{f\in H[\omega]_c}\|f(x)-S_n(f,x)\|_c=\frac{C^{(n)}[\omega]}{\pi}\log n+O\left(\omega\left(\frac{1}{n}\right)\right),$$

where $C^{(n)}[\omega]=\sup_{f\in H[\omega]_c}\left|\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx\right|$.

Xie Tingfan[1] considered the approximation of continuous function by its Cesáro means and proved the following

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Theorem B Suppose that $-1<\alpha\leqslant 0$ and $\varepsilon_n\downarrow 0$ as $n\to\infty$. If $f\in C_{2\pi}$ satisfies the one-side condition

$$f\left(x+\frac{2\pi}{2n+1+\alpha}\right)-f(x)\geqslant-\varepsilon_n \qquad (|x|\leqslant\pi,\ n=1,2,3,\cdots),$$

then

$$\|\sigma_n^\alpha(f)-f\|_c=O\left(\varepsilon_n\int_{\frac{1}{n}}^{\pi}\frac{dt}{n^\alpha t^{1+\alpha}}+\int_{\frac{1}{n}}^{\pi}\frac{\omega(f,t)}{n^{1+\alpha}t^{\alpha+2}}dt\right).$$

In the present peper, we consider the approximation of continuous function by its harmonic means, and establish the following theorems.

Theorem 1 Let $f(x)\in H[\omega]_c$. Then

$$\sup_{f\in H[\omega]_c}\|N_n(f,x)-f(x)\|_c=\frac{C^{(n)}[\omega]}{2\pi}\log n+O\left(\omega\left(\frac{1}{n}\right)\right),$$

where

$$C^{(n)}[\omega]=\sup_{f\in H[\omega]_c}\left|\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx dx\right|.$$

Theorem 2 Suppose that $\varepsilon_n\downarrow 0$ as $n\to\infty$. If $f\in C_{2\pi}$ satisfies the one-side condition

$$f\left(x+\frac{\pi}{n}\right)-f(x)\geqslant-\varepsilon_n \qquad (|x|\leqslant\pi,\ n=1,2,3,\cdots),$$

then

$$\|N_n(f,x)-f(x)\|_c=O\left(\varepsilon_n\log n+\frac{1}{n\log n}\int_{\frac{2\pi}{n}}^{\pi}\frac{\omega(f,t)}{t^2}\log\frac{1}{t}dt\right).$$

For the proof of theorem 1 we need the following lemmas.

Lemma 1 Let $D_\nu(t)$ be a Dirichlet kernel with order ν. Then we have

$$\sum_{\nu=0}^{n}\frac{D_\nu(t)}{n-\nu+1}=\frac{1}{2\sin\frac{t}{2}}\left[\log\frac{1}{2\sin\frac{t}{2}}\sin\left(n+\frac{3}{2}\right)t-\left(\frac{t}{2}-\frac{\pi}{2}\right)\cos\left(n+\frac{3}{2}\right)t\right]+O\left(\frac{1}{nt^2}\right), \qquad (0<t<\pi).$$

Lemma 2 If $f(x)\in C_{2\pi}$, then

$$N_n(f,x)-f(x)=\frac{1}{2\pi P_n}\int_{\frac{1}{n}}^{\pi}\frac{\varphi_x(t)}{2\sin\frac{t}{2}}\log\frac{1}{2\sin\frac{t}{2}}\sin\left(n+\frac{3}{2}\right)tdt+O\left(\omega\left(f,\frac{1}{n}\right)\right),$$

where $$\varphi_x(t)=f(x+t)+f(x-t)-2f(x).$$

The proof of theorem 2 depends on the following lemmas.

Lemma 3 Let $f\in C_{2\pi}$. Then

$$N_n(f,x)-f(x)=\frac{1}{\pi P_n}\sum_{\nu=3}^{2n_s}\frac{\log\frac{1}{2\sin t_\nu^*/2}}{t_\nu^*}(-1)^\nu\int_0^{\frac{2\pi}{n}}\{f(x+t_\nu^*+t)-f(x+t_\nu^*-t)$$

$$+f(x-t_\nu^*-t)-f(x-t_\nu^*+t)\}\sin nt\,dt+O\left(\omega\left(f,\frac{1}{n}\right)\right),$$

where
$$n_\nu=\max_{t_\nu\le\pi},\quad t_\nu^*=\frac{\nu}{n}\pi.$$

Lemma 4 If $f\in C_{2\pi}$, then

$$N_n(f,x)-f(x)=-\frac{1}{2}N_n(\Delta_n f,x)+O\left(\omega\left(f,\frac{1}{n}\right)\right),$$

where
$$\Delta_n f(x)=f\left(x+\frac{\pi}{n}\right)-f(x).$$

References

[1] 谢庭藩，杭州大学学报（自然科学报）3 (1979) 18-28.

[2] Ефимов, А. Б., Изв. АН, СССР, Серия матем., 23 (1959), 115-134.