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Globally Asymptotical Stability of Neutral Functional Differential Equations*

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§1 Introduction

Hale, J. K. obtained the result[1] of globally asymptotical stability for the autonomous retarded functional differential equations. In this paper we consider the globally asymptotical stability of general neutral functional differential equations and extend Hale's result.

§2 Definitions

Suppose r, τ are given real numbers with $r \geqslant 0$ and $\tau \in (-\infty, \infty)$, $\mathbb{R}^n$ is an n-dimensional linear vector space over the reals with norm $|\cdot|$, $C \overset{\text{def}}{=} C([-r,0], \mathbb{R}^n)$ is the Banach space of continuous functions mapping the interval $[-r,0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. We designate the norm of an element ϕ in C by $\|\phi\| = \sup\limits_{-r \leqslant \theta \leqslant 0} |\phi(\theta)|$ and let $x_t \in C$ be defined by $x_t(\theta) = x(t+\theta)$, $-r \leqslant \theta \leqslant 0$.

Consider the equation

$$\frac{dD(t)x_t}{dt} = f(t, x_t), \tag{1}$$

where f: $[\tau, \infty) \times C \to \mathbb{R}^n$ is continuous, satisfies a local Lipschitz condition with respect to ϕ in C and $f(t,0)=0$, for all $t \in [\tau, \infty)$; $D(t)\phi = \phi(0) - g(t,\phi)$, g: $[\tau,\infty) \times C \to \mathbb{R}^n$ is continuous, $g(t,\cdot)$ is bounded linear operator from C into $\mathbb{R}^n$ for each fixed t in $[\tau, \infty)$,

$$g(t,\phi) = \int_{-r}^{0} [d_\theta \mu(t,\theta)]\phi(\theta),$$

where $\mu(t,\cdot)$ is of bounded variation on $[-r,0]$. We also assume that g is uniformly nonatomic at zero; that is, there exists a continuous, nonnegative, nondecreasing functon $l(s)$ for s in $[0,r]$ such that

$$l(0)=0, \quad \left|\int_{-s}^{0} [d_\theta \mu(t,\theta)]\phi(\theta)\right| \leqslant l(s) \sup_{-s \leqslant \theta \leqslant 0} |\phi(\theta)|.$$

Suppose $C([\tau,\infty), \mathbb{R}^n)$ is the set of continuous functions mapping the interval $[\tau, \infty)$ into $\mathbb{R}^n$. For $(\sigma, \phi) \in [\tau,\infty) \times C$, $H \in C([\tau,\infty), \mathbb{R}^n)$, we let $x(\sigma,\phi,H)(t)$ be a solution of the equation

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$$\begin{cases} D(t)x_t = D(\sigma)\phi + H(t) - H(\sigma), & t \geqslant \sigma \\ x_\sigma = \phi. \end{cases} \tag{2}$$

We give some definitions.

Definition 1[2] Suppose $\mathscr{H} \subset C([\tau,\infty), \mathbb{R}^n)$. The operator $D(t)$ is said to be uniformly stable in relation to the set $\mathscr{H}$ if for any $\sigma \in [\tau,\infty)$, $\phi \in C$, $H \in \mathscr{H}$, there are positive numbers K, M, such that the solution $x(\sigma,\phi,H)$ of the equation (2) satisfies

$$\|x_t(\sigma,\phi,H)\| \leqslant K\|\phi\| + M \sup_{\sigma \leqslant u \leqslant t} |H(u) - H(\sigma)|, \quad t \geqslant \sigma.$$

Definition 2 The solution $x=0$ of $Eq.(1)$ is said to be globally asymptotically stable if it is stable and for every $(\sigma,\phi) \in [\tau,\infty) \times C$, the solution $x(\sigma,\phi)(t) \to 0$ as $t \to \infty$.

If v: $[\tau,\infty) \times C \to \mathbb{R}$ is continuous and $x(\sigma,\phi)$ is the solution of $Eq.(1)$ through (σ,ϕ), we define

$$\dot{V}_{(1)}(t,\phi) = \overline{\lim_{h \to 0^+}} \frac{1}{h}[v(t+h, x_{t+h}(t,\phi)) - v(t,\phi)].$$

§3 Theorems

We obtain the following results:

Theorem 1 Suppose the operator D(t) is uniformly stable in relation to $C([\tau,\infty), \mathbb{R}^n)$, and f: $[\tau,\infty) \times C \to \mathbb{R}^n$ takes $[\tau,\infty) \times$ (bounded sets of C) into bounded sets of $\mathbb{R}^n$. If there is a continuous function V: $[\tau,\infty) \times C \to \mathbb{R}$, $V(t, 0) = 0$, for all $t \in [\tau,\infty)$ such that

$$u(|D(t)\phi|) \leqslant V(t,\phi)$$

and

$$\dot{V}_{(1)}(t,\phi) \leqslant -v(|D(t)\phi|),$$

where v: $\mathbb{R}^+ \to \mathbb{R}^+$ is continuous function, u: $\mathbb{R}^+ \to \mathbb{R}^+$ is continuous nondecreasing function, $u(s)$ and $v(s)$ are positive for $s > 0$, and $u(s) \to \infty$ as $s \to \infty$, then the solution $x=0$ of $Eq.(1)$ is globally asymptotically stable.

If the operator $D(t)$ is independent of t, we have the following theorem.

Theorem 2 Suppose D, u, v, f, V are as in Theorem 1 and satisfy

$$u(|D\phi|) \leqslant V(t,\phi)$$

and

$$\dot{V}_{(1)}(t,\phi) \leqslant -v(|\phi(0)|).$$

Then the solution $x=0$ of $Eq.(1)$ is globally asymptotically stable.

Remark If $D(t)\phi$ (or $D\phi$) $= \phi(0)$, then Theorem 1 (or Theorem 2) becomes Hale's result. (See [1], Chap. 5, Corollary 3.1)

References

[1] Hale, J. K., Theory of functional differential equations, Springer—Verlay, New York, 1977;

[2] Cruz, M. A. and Hale, J. K., Stability of functional differential equations of neutral type. *J. Diff. Equns.* 7 (1970), 334—355.