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A Discussion on the Existence of Limit Cycle of Equations $\dot{x}=P(y)$, $\dot{y}=Q(x,y)$*

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For the system of differential equations having the form

$$\dot{x}=P(y), \qquad \dot{y}=Q(x,y), \tag{1}$$

R. M. Cooper has given an analytic criterion for the existence of limit cycle. [1] In which $P: R^1 \to R^1$ and $Q: R^2 \to R^1$ are everywhere continuous and locally Lipschitzian, $Q_y(x,y)$ is continuous at $(0,0)$ and is defined for all (x,y). His result is as follows:

Theorem. The system (1) has at least one stable limit cycle whenever (I) $yP(y)>0$ $(y\neq 0)$, $xQ(x,0)<0$ $(x\neq 0)$, (II) $\int_0^{\pm\infty} p(y)dy=+\infty$, $\int_0^{\pm\infty} Q(x,0)\,dx=-\infty$, (III) $Q_y(0,0)>0$, (IV) there exist numbers m, $a>0$ such that $Q_y(x,y)\leqslant -m<0$ for $|x|>a$, (V) $\frac{1}{P(y)}\sup\limits_{|x|\leq a} Q(x,y)=O^+(1)$ as $|y|\to+\infty$.

But this theorem is not true. In this paper we give a counterexample and some additional conditions to ensure the existence of limit cycle for system (1).

Example. $\dot{x}=y^{2k+3}$, $\dot{y}=Q(x,y)$, (2)

$$\text{where } Q(x,y)=\begin{cases} y+2y^{2k+1}-x^{2k+3} & \left(|x|<\sqrt{\frac{\pi}{2}}\right), \\ y\sin x^2+y^{2k+1}(1+\sin x^2)-x^{2k+3} & \left(\sqrt{\frac{\pi}{2}}\leqslant|x|\leqslant\sqrt{\frac{3\pi}{2}}\right), \\ -y-x^{2k+3} & \left(\sqrt{\frac{3\pi}{2}}<|x|\right) \end{cases}$$

Letting $a=\sqrt{\frac{3\pi}{2}}$, $m=1$, it is not difficult to show that the system (2) satisfies the conditions (I)—(V). Suppose it has a limit cycle L, then L is symmetric to $(0,0)$. Let $x_1=-x$, we turn over the part of L on the left half plane $x<0$ to the right half plane, and denote it by L', then L and L' both pass through same points $A(0,y_0)$ $(y_0>0)$ and $B(x_0,0)$. Let $\lambda(x,y)=\int_0^y P(y)dy-\int_0^x Q(x,0)dx$, in the vertical strip $|x|<\sqrt{\frac{\pi}{2}}$, $\frac{d\lambda}{dt}=y^{2k+4}(1+y^{2k})>0$ $(y\neq 0)$, so that $|x_0|>\sqrt{\frac{\pi}{2}}$. We denote

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the equations of the parts of L and L' between points A and B by $y=y_1(y)$ and $y=y_2(x)$, respectively. Hence $\frac{d\lambda}{dx}=Q[x,y_1(x)]-Q(x,0)$ on L, and $\frac{d\lambda_1}{dx}=-Q[-x,y_2(x)]+Q(-x,0)$ on L'.

It must be $\int_0^{x_0}\left(\frac{d\lambda}{dx}-\frac{d\lambda_1}{dx}\right)dx=0$, but $\int_0^{x_0}\left(\frac{d\lambda}{dx}-\frac{d\lambda_1}{dx}\right)dx=\int_0^{x_0}\{Q[x,y_1(x)]-Q[-x,y_2(x)]\}dx=\int_0^{\sqrt{\frac{\pi}{2}}}(y_1+y_1^{2k+1}+y_2+y_2^{2k+1})dx+\int_{\sqrt{\frac{\pi}{2}}}^{\sqrt{\frac{3\pi}{2}}}[(y_1+y_2)\sin x^2+(y_1^{2k+1}+y_2^{2k+1})(1+\sin x^2)]dx+\int_{\sqrt{\frac{3\pi}{2}}}^{x_0}(-y_1-y_2)dx>y_0^{2k+1}-2y_{max}-2y_{max}\cdot x_0$, in which $y_{max}=\max\limits_{0\le x\le x_0}y_1(x)$.

In the vertical strip $-\sqrt{\frac{\pi}{2}}<x<0$, it follows from (2) that $\frac{dy}{dx}=\frac{y+2y^{2k+1}-x^{2k+3}}{y^{2k+3}}>\frac{2}{y^2}$, and it implies that $y_0>\sqrt[3]{6}$.

Since $\frac{dy_1}{dx}=\frac{-y_1(x)-x}{[y_1(x)]^{2k+3}}<0$, $\frac{d\lambda}{dt}=-y_1^{2k+4}<0$ for $x>\sqrt{\frac{3\pi}{2}}$, and $\frac{dy_1(x)}{dx}<3$ for $0\leqslant x\leqslant\sqrt{\frac{3\pi}{2}}$, we have $y_{max}<y_0+3\sqrt{\frac{3\pi}{2}}$. Suppose the line $x=\sqrt{\frac{3\pi}{2}}$ intersected L at C, we have $\lambda(B)<\lambda(C)$, this means that $x_0^{2k+4}<y_0^{2k+4}+\sqrt{\frac{3\pi}{2}}^{2k+4}<(3y_{max})^{2k+4}$, hence $x_0<3y_{max}$.

If k is a sufficiently large integer, then

$$\int_0^{x_0}\left(\frac{d\lambda}{dx}-\frac{d\lambda_1}{dx}\right)dx>y_0^{2k+1}-8\left(y_0+3\sqrt{\frac{3\pi}{2}}\right)^2>0.$$

This contradiction shows that Cooper's theorem is not true. In his proof, he showed $\lambda_3-\lambda_1\to-\infty$ as $y_1\to+\infty$, and from this, he asserted that $|y_3|-|y_1|\to-\infty$ as $y_1\to+\infty$. But this assertion is not true. For example, let $P(y)=y$, then

$$\lambda_3-\lambda_1=\int_{y_1}^{y_3}y\,dy=\frac{1}{2}(|y_3|+y_1)(|y_3|-y_1)\to\infty \text{ as } y_1\to+\infty,$$

and it cannot follow that $|y_3|-y_1\to\infty$ as $y_1\to+\infty$.

Moreover, from $\lambda_5-\lambda_4<0$ it cannot be inferred that $|y_5|<|y_4|$, because P(y) is not odd function in general.

Under some additional conditions, we have proved the following Theorem. The system (1) has at least one stable limit cycle whenever (Ⅰ)—(Ⅴ) and (Ⅵ) there exist numbers $N_2>0>N_1$ such that $Q(x,N_1)>0$ $(a<x)$ and $Q(x,N_2)<0$ $(x<-a)$, or (Ⅵ)′ $\frac{P(y)}{y}=O^+(1)$ as $|y|\to\infty$.

In particular, if $Q(x,0)$ is bounded, then it follows from (Ⅰ)—(Ⅴ) that the condition (Ⅵ) is satisfied.

Reference

[1] Cooper, R. M, *J. Math. Anal. and Appl*, 34 (1971), 412-417.