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On Representing the General Solution with the Special Solutions for the Differential Equation $y'=\sum_{i=0}^{n}a_i(x)y^i$*

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It is well known that the $n+1$ coefficients of the equation

$$y'=a_n(x)y^n+a_{n-1}(x)y^{n-1}+\cdots+a_1(x)y+a_0(x) \tag{1}$$

can be completely determined by any $n+1$ different special solutions of the same equation. Hence, any solution of the same equation can be completely determined by its initial value and the $n+1$ special solutions. In addition, when $0\leqslant n\leqslant 2$, the general solution of Eq. (1) can be represented with $n+1$ different special solutions and an integral constant, and the representation is independent of the concrete forms of the coefficients $a_i(x)$. Therefore, we give

Definition 1 For a definite non negative integer n, the function $F_n(y_1,y_2,\cdots,y_{n+1},C)$, which is independent of the coefficients $a_i(x)$, $i=0,1,2,\cdots,n$, is called the representative function of the general solution of Eq. (1), if the general solution of Eq. (1) can be represented by

$$y(x)=F_n(y_1(x),y_2(x),\cdots,y_{n+1}(x),C) \tag{2}$$

where $y_i(x)$, $i=1,2,\cdots,n+1$, are any $n+1$ different special solutions of the same equation, and C is the integral constant.

In this connection, we deduce

Theorem 1 If $F_n(y_1,\cdots,y_{n+1},C)$ is a representative function of the general solution, then, $y=F_n(y_1,\cdots y_{n+1},C)$ is the general solution of the following first order partial differential equation system

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ y_1 & y_2 & \cdots & y_{n+1} \\ y_1^2 & y_2^2 & \cdots & y_{n+1}^2 \\ \vdots & \vdots & & \vdots \\ y_1^n & y_2^n & \cdots & y_{n+1}^n \end{pmatrix} \begin{pmatrix} \dfrac{\partial y}{\partial y_1} \\ \dfrac{\partial y}{\partial y_2} \\ \vdots \\ \dfrac{\partial y}{\partial y_{n+1}} \end{pmatrix} = \begin{pmatrix} 1 \\ y \\ y^2 \\ \vdots \\ y^n \end{pmatrix} \tag{3}$$

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and vice versa.

By studying the consistency of the system (3), we obtain

Conclusion 1 When $n\geqslant 3$, there exists no representative function of the general solution.

Consider a more general problem. Let $a_1<a_2<\cdots<a_n$ are n definite real numbers. We give

Definition 2 The function $F_{a_1,\cdots;a_n}(y_1,\cdots,y_n,C)$ is called the representative function of the general solution of the equation

$$y'=a_1(x)y^{a_1}+a_2(x)y^{a_2}+\cdots+a_n(x)y^{a_n} \tag{4}$$

if the form of the function is independent of the coefficients $a_i(x)$, $i=1,2,\cdots,n$, and if the general solution of Eq. (4) can be represented by

$$y(x)=F_{a_1,\cdots;a_n}(y_1(x),\cdots,y_n(x),C) \tag{5}$$

where $y_i(x)$, $i=1,2,\cdots,n$, are any n different special solutions of the same equation, and C is the integral constant.

Using the previous method, we obtain the following conclusions:

Conclusion 2 When $n=1$, if $a_1=1$, then $F_{a_1}(y_1,C)=Cy_1$, and if $a_1\neq 1$, then $F_{a_1}(y_1,C)=(y_1^{1-a_1}+C)^{\frac{1}{1-a_1}}$.

Conclusion 3 When $n=2$, if and only if one of a_1 and a_2 is equal to 1, there exists a representative function of the general solution, $F_{a_1;a_2}(y_1,y_2,C)=[y_1^{1-a}+C(y_2^{1-a}-y_1^{1-a})]^{\frac{1}{1-a}}$, where a is equal to one of a_1 and a_2 which is not equal to 1. The corresponding equation is just the Bernoulli equation.

Conclusion 4 When $n=3$, if and only if $a_2=1$ and $a_1+a_3=2$, there exists a representative function of the general solution, $F_{a_1;a_2;a_3}(y_1,y_2,y_3,C)=\{z_1+[(z_2-z_1)^{-1}+C((z_3-z_1)^{-1}+(z_2-z_1)^{-1})]^{-1}\}^{\frac{1}{1-a_1}}$, where $z_i=y_i^{1-a_1}$, $i=1,2,3$. By the transformation $z=y^{1-a_1}$, the corresponding equation can be transformed into the Riccati equation.

Conclusion 5 When $n\geqslant 4$, for any different $a_1,\cdots,a_n$, there exists no representative function of the general solution.

The conclusions obtained show that the Bernoulli equation and the Riccati equation occupy special positions in the equations with form (4).