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The Algebraic Critical Cycle and Bifurcation Function of Limit Cycle for the System $\dot{x}=xy,\ \dot{y}=\sum\limits_{0<j+i<2}a_{ij}x^iy^j$*

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In the quadratic system possessing one integral straight line

$$\dot{x}=xy,\ \dot{y}=\sum_{0<i+j<2}a_{ij}x^iy^j \tag{1}$$

there may exist a triangle critical cycle which is formed of two rays and a section of arc of equator on Poincaré sphere, if (1) possesses two integral straight lines[1]. Besides, in that paper the bifurcation function of limit cycle of (1) is also found. In this article we prove that there may exist a lune of critical cycle which is formed of an algebraic curve and a section of arc of the equator on the Poincaré sphere; or is formed of an algebraic curve and a straight line. Moreover, we will also find out bifurcation function of limit cycle.

In this paper we always suppose that (1) possesses four singular points. Without loss of generality, we may regard them as: $P_1(1,0)$,$P_2\left(\frac{c}{a},0\right)$, $P_3(0,1)$, $P_4\left(0,\frac{c}{b}\right)$.

Thereupon, (1) can be written in the form $E(\alpha)$:

$$\dot{x}=xy,\ \dot{y}=c-(a+b)x-(b+c)y+ax^2+(b+c)xy+by^2+\alpha xy. \tag{$E(\alpha)$}$$

It is clear that $E(\alpha)$ forms a rotated vector field which depends on α.

We also suppose that $P_1(1,0)$ is a focal point whose focal quantities are equal to

$$\bar{V}_1=\alpha \quad \bar{V}_3=\frac{1}{(c-a)^{\frac{5}{2}}}(b+c)[a+b(c-a)]$$

If $\bar{V}_1=\bar{V}_3=0$, $P_1(1,0)$ is vortex point;

Theorem 1 Let $E(\alpha)$ satisfy: $c=b-\frac{1}{2}$, $c>\frac{1}{2}$, $a<0$, then for $\alpha=\bar{\alpha}=-\frac{1}{2}$; $E(\alpha)$ possesses an algebraic critical cycle around the singular point $P_1(1,0)$; The critical cycle is formed of a branch of the hyperbola

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$$F(x,y)=ax^2+2cxy+(b-1)y^2-\frac{c+a(b-1)}{c}x-2(b-1)y+(b-1)=0 \qquad (3)$$

and a section of arc of the equator on the Poincaré sphere.

Moreover, the necessary and sufficient condition that $E(\alpha)$ possesses just one limit cycle around the singular point $(1,0)$ is $-\frac{1}{2}<\alpha<0$.

Proof From $c>\frac{1}{2}$, $a<0$, we have $\overline{V}_3>0$. According to the theory of rotated vector field we imply that $E(\alpha)$ possesses unique limit cycle as $-1\ll\alpha<0$. Moreover, from $c=b-\frac{1}{2}$, $a=-\frac{1}{2}$, we state that $E(x)$ possesses a hyperbola solution (3), therefore, $E\left(-\frac{1}{2}\right)$ does not posses limit cycle. We now prove that the right branch of (3) and a section of arc of equator on Poincaré sphere forms a lune of critical cycle. The proof is divided into two steps: i) the infinite points of the hyperbola (3) are saddle points of $E\left(-\frac{1}{2}\right)$ and there are not any singular points of $E\left(-\frac{1}{2}\right)$ on the preceding lune except these two saddle points. ii) There is uuique singular point $P_1(1,0)$ inside the preceding lune.

i) Having performed simple calculation, we know that $E\left(-\frac{1}{2}\right)$ possesses three infinite singular points. They are $N(0,1,0)$, $N_i(k_i,1,0)(i=1,2)$, where k_i are two roots of equation $aK^2+2cK+c-\frac{1}{2}=0$. Moreover, N is a nodal point. N_i are saddles on the hyperbola (3).

The intersection points that the hyperbola (3) intersects with the coordinate axes are $(0,1)$, $\left(-\frac{c}{a},0\right)$ and $\left(\frac{c-\frac{1}{2}}{c},0\right)$. The front two of them are singular points of $E\left(-\frac{1}{2}\right)$ and are located at the left branch of the hyperbola (3). Consequently, on the right branch of the hyperbola (3), there are not any finite singular points of $E\left(-\frac{1}{2}\right)$. Moveover, N $(0,1,0)$ is located neither at the hyperbola (3) nor at arcs of the equator on Poincaré sphere, but between branches of hyperbola (3).

ii) Paying attention to $0<\frac{c-\frac{1}{2}}{2}<1$, we find that the state is true.

Finally we prove the necessary and sufficient condition for $E(\alpha)$ to possess just one limit cycle around $P_1(1,0)$ is $-\frac{1}{2}<\alpha<0$,

Based on rhe theory of rotated vector field the necessity is true. We now prove sufficiency. $P_1(1,0)$ is a stable singular point. In accordance with the theory of rotated vector feild, the critical cycle of $E\left(-\frac{1}{2}\right)$ is outer Bendixson bound of $E(\alpha)$ and the trajectory of $E(\alpha)$ cross this bound from inside to outside. Therefore, $E(\alpha)$ possesses limit cycle. Moveover, since $E(\alpha)$ possesses one straight solution, the limit cycle is unique. The proof of theorem 1 is completed.

Theorem 2 Let $E(a)$ satisfy $(b+c)^2-4ab+2a=0$, $b<0$, $c>0$, then for $a=\bar{a}=\frac{b+c}{2(1-2b)}$, $E(a)$ possesses an algebraic critical cycle around the singular point $P_1(1,0)$. The critical cycle is formed of the parabola

$$\left(x+\frac{b+c}{2a}y\right)^2-\frac{4bc+(b+c)^2}{4ab}x-\frac{(b+c)^3}{4a^2b}y+\frac{c(b+c)^2}{4a^2b}=0$$

and a straight segment between the singular point $P_3(0,1)$ and singular $P_4\left(0,\frac{c}{b}\right)$.

Moreover, in order that $E(a)$ possesses just one limit cycle around the singular point $(1,0)$, the necessary and sufficient condition is $0<a<\frac{b+c}{2(1-2b)}$ as $(b+c)>0$, or $\frac{b+c}{2(1-2b)}<a<0$ as $(b+c)<0$.

(The proof of theorem 2 is parallel to that of theorem 1).