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Galerkin Methods for Coupled Hyperbolic and Parabolic Linear System*

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In this paper, we consider the Galerkin methods for the coupled hyperbolic and parabolic linear system and obtain *a priori* error estimates for semi-discrete and fully discrete Galerkin approximations on the basis of paper [1], but we release the restriction of weak coupled action of the solution [1].

§1 Introduction

In [1], we considered the coupled system arising from thermo-elastic problems

$$\begin{cases} u_{tt}-a_1u_{xx}+\varepsilon b_1v_{tt}=f(x,t), \\ v_t-a_2v_{xx}+\varepsilon b_2u_t=g(x,t), \quad (x,t)\in\Omega\times[0,T], \quad \Omega=[0,1]. \end{cases}$$

Here u, v denote the displacement and temperature respectively.

We assume that $\varepsilon^2<1/b_1b_2$ in order to obtain error estimates, this restriction means the weak coupled action of the solution. In order to release this restriction, we consider the coupled system about stress and temperature instea',

$$\begin{cases} u_{tt}-\nabla(a_1(x)\nabla u)+b_1\sum_{i=1}^{n}v_{x_i}=f(x,t), \\ v_t-\nabla(a_2(x)\nabla v)+b_2\sum_{i=1}^{n}u_{tx_i}=g(x,t), \quad (x,t)\in\Omega\times[0,T] \end{cases} \tag{1.1}$$

$$\begin{cases} u(x,t)=v(x,t)=0, \quad (x,t)\in\partial\Omega\times[0,T] \\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad v(x,0)=v_0(x), \quad x\in\Omega, \end{cases} \tag{1.2}$$

where $\Omega\subset R^n$ is a bounded polygonal domain. We assume that there are constants d_0, c_0 such that $0<d_0\leqslant a_i(x)\leqslant c_0$, $x\in\Omega$ for $a_i\in C^0(\bar\Omega)$, $i=1,2$; b_1 and b_2 are positive constants;

We give a family $\{S_h\}$ of finite dimensional subspaces of $H_0^1(\Omega)$ such that for some integar $r\geqslant 2$ and small h,

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$$\inf_{y\in v_h}\{\|v-y\|+h\|\nabla(v-y)\|\}\leqslant ch^s\|v\|_s,\quad 1\leqslant s\leqslant r,\quad \text{for } v\in H^s(\Omega)\cap H_0^1(\Omega).$$

$\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the inner product and norm of $L^2(\Omega)$, and

$$L^q(T,H^k)=\{u(x,t)\in H^k,\ 0\leqslant t\leqslant T,\ \|u(t)\|_k\in L^q([0,T])\}.$$

§2 Semi-discrete approximation

A weak form of the system (1.1) and (1.2) is

$$\begin{cases}\langle u_{tt},y\rangle+\langle a_1\nabla u,\nabla y\rangle+b_1\langle\nabla v,y\rangle=\langle f,y\rangle,\\ \langle v_t,z\rangle+\langle a_2\nabla v,\nabla z\rangle+b_2\langle\nabla u_t,\nabla z\rangle=\langle g,z\rangle,\quad \text{for } y,z\in H_0^1(\Omega),\\ \langle u(x,0)-u_0(x),y\rangle=0,\langle u_t(x,0)-{}_1u(x),y\rangle=0,\langle v(x,0)-v_0(x),y\rangle=0,y\in H_0^1.\end{cases}\tag{2.1}$$

Semi-discrete approximations $U(\cdot,t)$, $V(\cdot,t)\in S_h$, $t\in[0,T]$ are defined by the following equations:

$$\begin{cases}\langle U_{tt},y\rangle+\langle a_1\nabla U,\nabla y\rangle+b_1\langle\nabla V,y\rangle=\langle f,y\rangle,\quad y\in S_h,\\ \langle V_t,z\rangle+\langle a_2\nabla V,\nabla z\rangle+b_2\langle\nabla U_t,z\rangle=\langle g,z\rangle,\quad z\in S_h,\end{cases}\tag{2.2}$$

where $U(x,0)$, $U_t(x,0)$ and $V(x,0)\in S_h$ are proper approximations of u_0, u_1 and v_0 in S_h, $\langle\nabla v,y\rangle\equiv\sum_{i=1}^{n}\langle v_{x_i},y\rangle$.

Assume that u, v are the solutions of (1.1) and (1.2), we introduce elliptic projections $W,M\in S_h$ of u and v:

$$\begin{cases}\langle a_1\nabla W,\nabla y\rangle=\langle a_1\nabla u,\nabla y\rangle,\quad y\in S_h,\\ \langle a_2\nabla M,\nabla z\rangle=\langle a_2\nabla v,\nabla z\rangle,\quad z\in S_h,\quad t\in[0,t].\end{cases}\tag{2.3}$$

We set $\eta=W-u$, $\sigma=M-v$, $\xi=U-W$, $\zeta=V-M$. Then ξ,ζ satisfy

$$\begin{cases}\langle\xi_{tt},y\rangle+\langle a_1\nabla\xi,\nabla y\rangle+b_1\langle\nabla\zeta,y\rangle=-\langle\eta_{tt}+b_1\nabla\sigma,y\rangle,\quad y\in S_h,\\ \langle\zeta_t,z\rangle+\langle a_2\nabla\zeta,\nabla z\rangle+b_2\langle\nabla\xi_t,z\rangle=-\langle\sigma_t+b_2\nabla\eta_t,z\rangle,\quad z\in S_h.\end{cases}\tag{2.4}$$

It is easy to obtain the following result by means of standard arguments.

Theorem 1: Let u, v, u_t, $v_t\in L^\infty(T,H^k)$, $u_{tt}\in L^2(T,H^k)$ and U,V be the solutions of (2.2). Suppose $\|U(0)-W(0)\|_1+\|U_t(0)-W_t(0)\|+\|V(0)-M(0)\|=O(h^k)$. Then there exists a constant c such that

$$\|U_t-u_t\|_{L^\infty(T,L^2)}+\|U-u\|_{L^\infty(T,L^2)}+\|V-v\|_{L^\infty(T,L^2)}\leqslant ch^{k-1},\quad 2\leqslant k\leqslant r.$$

§3 Fully discrete approximation

In this section we discuss the fully discrete Galerkin approximation of (2.2).

Let Δt be the time step, $t_j=j\Delta t$. We denote $\varphi_j=\varphi(x,t_j)$ for any continuous function φ, and we define $\varphi_{j+\frac12}=(\varphi_{j+1}+\varphi_j)/2$, $\varphi_{j,\theta}=\frac12(1+\theta)\varphi_{j+1}+\frac12(1-\theta)\varphi_j$, $\varphi_{\theta,j}=\theta\varphi_{j+1}+(1-2\theta)\varphi_j+\theta\varphi_{j-1}$, $\partial_t\varphi_{j+\frac12}=(\varphi_{j+1}-\varphi_j)/\Delta t$, $\partial_t^2\varphi_j=(\varphi_{j+1}-2\varphi_j+\varphi_{j-1})/(\Delta t)^2$ and $\partial_t\varphi_{j+\frac12,\theta}=\frac12(1+\theta)\partial_t\varphi_{j+\frac12}+\frac12(1+\theta)\partial_t\varphi_{j-\frac12}$, $\theta\in[0,1]$.

We define that the fully discrete Galerkin approximation $\{U_j,V_j\}_0^N\subset S_h$ of (2.2) is a solution of the following system of equations:

$$\begin{cases}\langle\partial_t^2 U_j,y\rangle+\langle a_1\nabla U_{\frac{1}{4},j},\nabla y\rangle+b_1\langle\nabla V_{\frac{1}{4},j},y\rangle=\langle f_{\frac{1}{4},j},y\rangle, & y\in S_h.\\ \langle\partial_t V_{j+\frac{1}{2}},z\rangle+\langle a_2\nabla V_{j,0},\nabla z\rangle+b_2\langle\nabla\partial_t U_{j+\frac{1}{2}},z\rangle=\langle g_{j,0},z\rangle, & z\in S_h.\end{cases}\tag{3.1}$$

where U_0,U_1 and V_0 are suitable ones

Using the standard argument for error estimates, we have the following result:

Theorem 2 Let $u,v,u_t,v_t\in L^\infty(T,H^k)$, u_{tt}, u_{tttt}, $u_{ttt}\in L^2(T,L^2)$ and $u_{tt}\in L^2(T,H^1)$. If $\|\partial_t\xi_{\frac{1}{2}}\|+\|\xi_{\frac{1}{2}}\|=O(h^k+(\Delta t)^2)$, $\zeta_0=0$, $h^{-1}\Delta t$ is bounded, then there exists a constant C independent of $h,\Delta t$, u and v such that

$$\|\partial_t(U-u)\|_{\bar{L}^\infty(N,L^2)}+\|U-u\|_{\bar{L}^\infty(N,H^1)}+\|V-v\|_{\bar{L}^\infty(N,L^2)}\leqslant C(h^{k-1}+(\Delta t)^2),$$

where $\|\varphi\|_{\bar{L}^\infty(N,H^k)}=\max\limits_{1\le j\le N-1}\|\varphi_{j+\frac{1}{2}}\|_k.$

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