第3卷 第1期
1983年 1月

数学研究与评论
JOURNAL OF MATHEMATICAL RESEARCH AND EXPOSITION

Vol.3 No.1
Jan. 1983

# Some Notes on Solvability of LPDO*

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The purpose of this paper is to discuss how the lower order terms of a LPDO with multiple characteristics influence the solvability of the operators. We begin with the followiog operator:

$$L(x,D)=D_1^2+x_1^2D_2^2+pD_2+qD_1+c$$

where $p,q,c$ are assumed be complex constants.

**Theorem 1** The operator $L$ will be locally solvable for any $q,c$, if either $|\mathrm{Re}\ p|<1$ or $\mathrm{Im}\ p=0$.

**Proof** (i) Suppose $|\mathrm{Re}\ p|<1$. Let $\omega$ be a neighborhood of the zero in $R^2$. Let $\varphi\in C_0^\infty(\omega)$. It is clear that

$$\begin{aligned}|(L\varphi,\varphi)| &= |((1-|\mathrm{Re}\,p|)(D_1^2+x_1^2D_2^2)\varphi,\varphi)+|\mathrm{Re}\,p|\cdot((D_1^2+x_1^2D_2^2+\mathrm{Sign\ Re}\,p\cdot D_2)\varphi,\varphi)\\ &+i\,\mathrm{Im}\,p(D_2\varphi,\varphi)+q(D_1\varphi,\varphi)+c(\varphi,\varphi)|\geqslant|((1-\mathrm{Re}\,p)(D_1^2+x_1^2D_2^2)\varphi,\varphi)+|\mathrm{Re}\,p|\\ &\cdot((D_1^2+x_1^2D_2^2+\mathrm{Sign\ Re}\,p)\varphi,\varphi)+i\,\mathrm{Im}\,p(D_2\varphi,\varphi)|-|q(D_1\varphi,\varphi)|-|c(\varphi,\varphi)|. \quad (1)\end{aligned}$$

Note that

$$((D_1^2+x_1^2D_2^2)\varphi,\varphi)=\|D_1\varphi\|^2+\|D_2(x_1\varphi)\|^2\geqslant\|D_1\varphi\|^2 \quad (2)$$

and that

$$((D_1^2+x_1^2D_2^2+\mathrm{Sign\ Re}\,p\cdot D_2)\varphi,\varphi)=\begin{cases}\|D_1+ix_1D)\varphi\|^2 & \text{for Re } p>0,\\ 0 & \text{for Re } p=0,\\ \|(D_1-ix_1D_2)\varphi\|^2 & \text{for Re } p<0.\end{cases}$$

It follows that

$$|\mathrm{Re}\,p|\,((D_1^2+x_1^2D_2^2+\mathrm{Sign\ Re}\,p\cdot D_2)\varphi,\varphi)\geqslant 0. \quad (3)$$

Since $(D_2\varphi,\varphi)=(\varphi,D_2\varphi)=\overline{(D_2\varphi,\varphi)}$ hence $\mathrm{Im}(D_2\varphi,\varphi)=0$, namely, $(D_2\varphi,\varphi)$ is real. From this and (1), (2), (3), we obtain:

*Received Dec. 30, 1981.

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$$|(L\varphi,\varphi)| \geqslant \sqrt{(1-\operatorname{Re} p)^2\|D_1\varphi\|^2+|\operatorname{Im} p(D_2\varphi,\varphi)|^2}-|q(D_1\varphi,\varphi)|^2-|c|\,\|\varphi\|^2 \quad (4)$$

Let $\varepsilon=\max|x_1'-x_2'|$, $(x_1',x_2')\in\omega$, $(x_1'',x_2'')\in\omega$. It is easy to see that $\|D_1\varphi\|\geqslant\frac{1}{\varepsilon}\|\varphi\|$, $\forall\varphi\in C_0^\infty(\omega)$. Making ues of Cauchy's inequality, we obtain $\|L\varphi\|\geqslant c_0\|\varphi\|$, for some constant $c_0>0$, $\forall\varphi\in C_0^\infty(\omega)$, which implies the local solvability of the operator L.

(ii) Suppose $\operatorname{Im} p\neq 0$. Denote $A=\|D_1\varphi\|^2+\|x_1D_2\varphi\|^2+P(D_2\varphi,\varphi)$. Note that $(D_2\varphi,\varphi)$ is real, as has been shown. We obtain

$$|A|\equiv\sqrt{(\|D_1\varphi\|^2+\|x_1D_2\varphi\|^2+\operatorname{Re} p(D_2\varphi,\varphi))+|\operatorname{Im} p(D_2\varphi,\varphi)|^2}$$

$$\geqslant\|D_1\varphi\|^2+\|x_1D_2\varphi\|^2-|\operatorname{Re} p|\cdot|(D_2\varphi,\varphi)|$$

and $|A|\geqslant|\operatorname{Im} p|\cdot|(D_2\varphi,\varphi)|$. It follows that $|A|\geqslant\frac{|\operatorname{Im} P|}{|\operatorname{Im} p|+|\operatorname{Re} p|}\|D_1\varphi\|^2$. Thus, we have:

$$|(L\varphi,\varphi)|\geqslant|A|-|q(D_1\varphi,\varphi)|-|c|\cdot\|\varphi\|^2$$

$$\geqslant\frac{|\operatorname{Im} p|}{|Re\ p|+|Im\ p|}\|D_1\varphi\|^2-|q|\cdot\|D_1\varphi\|\cdot\|\varphi\|-|c|\cdot\|\varphi\|^2$$

Applying the same argument as in the case (i) we obtain $\|L\varphi\|\geqslant c_0\|\varphi\|^2$, for some constant $c_0>0$ and for every $\varphi\in C_0^\infty(\omega)$, provided that the size of $\omega$ is chosen to be sufficiently small. Thus, the proof of "theorem 1" has been completed.

It is easy to see that "theorem 1" can also be proved when $p, q, c$ are functions belonging to $L_\infty(\omega_0)$, where $\omega_0$ is a neighborhood of zero. This theorem shows that the lower order terms $qD+c$ do not have any influence on the solvability of $L$, when either $|\operatorname{Re} p|<1$ or $\operatorname{Im} p\neq 0$.

On the other hand, it is well known that the operator $D_1^2+x_1^2D_2^2+D_2$ is nonsolvable[2]. If we add the lower order term $qD+C$ to this operator, what influence will they have on the solvability? To answer this question, we give the following:

**Theorem 2** Let $L(x,D)=D_1^2+x_1^2D_2^2\pm D_2+C$, namely, $p=\pm1$, $q=0$. If either $\operatorname{Re} c>0$ or $\operatorname{Im} c\neq 0$, then $L(x,D)$ is locally solvable.

**Proof** $|(L\varphi,\varphi)| = |\|(D_1\pm ix_1D_2)\varphi\|^2+\operatorname{Re} c\|\varphi\|^2+i\operatorname{Im} c\|\varphi\|^2|$

$$=\sqrt{(\|(D_1\pm ix_1D_2)\varphi\|^2+\operatorname{Re} c\|\varphi\|)^2+(\operatorname{Im} c\|\varphi\|^2)^2}$$

$$\geqslant\max(|\operatorname{Im} c|,\operatorname{Re} c+|\operatorname{Im} c|)\|\varphi\|^2=c_0\|\varphi\|^2$$

From the conditions given to $c$, it is obvious that $\max(|\operatorname{Im} c|,\operatorname{Re} c)>0$, $c_0>0$. Therefore we obtain:

$$\|L\varphi\|\geqslant c_0\|\varphi\| \qquad \forall\varphi\in C_0^\infty(\omega).$$

Hence the proof is completed.

It follows from "theorem 2" that the lowest order term $c$ can play a crucial role in the solvability of L. After the change of unknown function $v=e^{i\frac{q}{2}x_1}u$ and by "theorem 2", we obtain the following.

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**Theorem 3** The operator $D_1^2+x_1^2\ D_2^2\pm D+qD_1+c$ is locally solvable, if either $\mathrm{Re}\left(-\frac{q^2}{4}+c\right)>0$ or $\mathrm{Im}\left(-\frac{p}{4}+c\right)\neq 0$.

In the same way as in the proof of theorem 2, we can also obtain the following.

**Theorem 4** Let $L(x,D)=L_0(x,D)+R(x,D)+iQ(x,D)+L_1(x,D)$. Suppose that

(a) $(L_0\varphi,\varphi)\geqslant 0,\qquad \forall\varphi\in C_0^\infty(\omega)$;

(b) either $(R\varphi,\varphi)\geqslant c'\|\varphi\|^2$ for some constant $c'>0$, or $|(Q\varphi,\varphi)|\geqslant c''\|\varphi\|^2$ for some constant $c''>0$ and $\mathrm{Im}(Q\varphi,\varphi)=0$;

(c) there is a constant $c_1<1$ such that

$$(L_1\varphi,\varphi)\leqslant c_1\ |((L_0+R+iQ)\varphi,\varphi)|.$$

Then the operator $L(x,D)$ is locally solvable. In particular, choose $L=A^tA+c$, where $A$ is a nonsolvable operator and c is a constant.

Note that the $A^tA$ is nonsolvable when $c=0$, but $L$ is solvable when either $\mathrm{Re}\ c>0$ or $\mathrm{Im}\ c\neq 0$. Thus, we have seen that the lowest order term $c$ may have a crucial influence on the solvability of $L$.

Acknowledgement:

We thank Professor C. Y. Chen (陈庆益) for his interest and encouragement.

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