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Cubic L-Spline Interpolation at a Biinfinite Knot Sequence

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Let $P(D)=(D^2-\alpha^2)(D^2-\beta^2)$, where $D=\dfrac{d}{dx}$, $\beta\geqslant\alpha\geqslant 0$, and

$$\pi(p)=\{f(x)\in C^4_{(-\infty,+\infty)};p(D)f\equiv 0\}.$$

Let further $\Delta=(t_i)_{-\infty}^{\infty}$ be a strictly increasing sequence with $t_{\pm\infty}=\lim\limits_{i\to\pm\infty}t_i$. We denote $\mathbf{Z}$ be the set of all integers, $\mathbf{R}$ be the set of all real numbers. Let

$$S_p(\Delta)=\{s(x)\in C^2_{(t_{-\infty},t_{+\infty})};s|_{(t_\nu,t_{\nu+1})}\in\pi(P),\nu\in\mathbf{Z}\},$$
$$Y=\{y=(y_\nu)_{-\infty}^{\infty};y_\nu\in\mathbf{R},(y_\nu)\in L^\infty\}.$$

Problem A For an arbitarily given $y=(y_\nu)\in Y$, what conditions should Δ satisfy for existing an unique $s(x)\in S_p(\Delta)\cap L^\infty_{(t_{-\infty},t_{+\infty})}$ such that $s(t_\nu)=y_\nu,\nu\in\mathbf{Z}$?

Firstly we assume that $\beta>\alpha>0$, hence $e^{\alpha x}$, $e^{-\alpha x}$, $e^{\beta x}$, $e^{-\beta x}$ form a base of $\pi(P)$. Let us compose $p_j(x)\in\pi(P)$, $j=0,1,2,3$, satisfying

$$p_j^{(i)}(0)=\delta_{j,i},\ p_j(1)=0,\ i,j=0,1,2,$$
$$p_3^{(i)}(0)=0,\quad p_3(1)=0,\ i=0,1,2,$$

and denote $\Delta t_i=t_{i+1}-t_i$, $m_i=\Delta t_i/\Delta t_{i-1}$

$$c=(\beta\mathrm{sh}\alpha\cdot\mathrm{ch}\beta-\alpha\mathrm{sh}\beta\cdot\mathrm{ch}\alpha)/(\alpha\cdot\mathrm{sh}\beta-\beta\mathrm{sh}\alpha),$$
$$f=(\beta^2-\alpha^2)\mathrm{sh}\beta\cdot\mathrm{sh}\alpha/(\alpha\mathrm{sh}\beta-\beta\mathrm{sh}\alpha),$$
$$d=[(\alpha^2+\beta^2)\mathrm{sh}\beta\mathrm{sh}\alpha-2\alpha\beta\mathrm{ch}\beta\cdot\mathrm{ch}\alpha+2\alpha\beta]/(\beta^2-\alpha^2)(\alpha\mathrm{sh}\beta-\beta\mathrm{sh}\alpha),$$

$$\begin{cases} A=\begin{pmatrix} c & d\\ f & c\end{pmatrix},\ D(m)=\begin{pmatrix} m & 0\\ 0 & m^2\end{pmatrix},\ A(m)=A\cdot D(m), & \\ x_{i,l}=(-1)^i(\Delta t_{i-1})^l s^{(l)}(t_i), & l=1,2,\ i\in\mathbf{Z}\\ b_{i,l}=(-1)^{i+1}(y_i p_3^{(l)}(1)+y_{i+1}p_3^{(l)}(1), & l=1,2,\ i\in\mathbf{Z}\\ x_i=(x_{i,1},x_{i,2})^T,b_i=(b_{i,1},b_{i,2})^T, & i\in\mathbf{Z}.\end{cases}\tag{1}$$

Then the solvability of problem A is equivalent to whether the difference equation

$$x_{i+1}=A(m_i)x_i+b_i,\quad i\in\mathbf{Z}\tag{2}$$

has a bounded solution.

Theorem 1 If $m_i=m^*=\frac{1}{2}(c+1+\sqrt{(1+c)^2-4})$, $i\in\mathbf{Z}$, then there exists a non-trivial $s(x)\in S_p(\Delta)\cap L^\infty_{(t_{-\infty},t_{+\infty})}$, which satisfies $s(t_i)=0$. Similarly if $m_i=(m^*)^{-1}$,

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$i\in \mathbf{Z}$, then we have the same conclusion.

Theorem 2 Let $m=\sup\limits_{|i-j|=1} \Delta t_i/\Delta t_j$. If

$$m<m^*=\frac{1}{2}(c+1+\sqrt{(1+c)^2-4}), \tag{3}$$

then problem A has an unique bounded solution.

To prove Theorem 2, we need some lemmas.

Lemma 1 Under the condition of theorem 2, the homogeneous equation

$$x_{i+1}=A(m_i)x_i,\ i\in \mathbf{Z} \tag{4}$$

has no bounded non-trivial solutions.

For a matrix $R=(r_{ij})$, we denote $|R|=(|r_{ij}|)$.

Lemma 2 If $\frac{1}{m}\leqslant m_i\leqslant m$, $i\in \mathbf{Z}$, and for $i,n\in \mathbf{Z}$, let

$$\begin{pmatrix} s_n^i & l_n^i \\ p_n^i & q_n^i \end{pmatrix}=A(m_{i+n-1})\cdots A(m_i),\begin{pmatrix} \bar{s}_n^i & \bar{l}_n^i \\ \bar{p}_n^i & \bar{q}_n^i \end{pmatrix}=|A^{-1}(m_{i-n})|\cdots|A^{-1}(m_{i-1})|,$$

then there exist $\alpha_i>0$, $\beta_i>0$, such that

$$\frac{s_n^i}{l_n^i}>\alpha_i>\frac{p_n^i}{q_n^i},\quad \frac{\bar{s}_n^i}{\bar{l}_n^i}>\beta_i>\frac{\bar{p}_n^i}{\bar{q}_n^i},\quad n=1,2,\cdots,$$

furthermore, there exist positive numbers $\eta,\bar{\eta},\zeta,\bar{\zeta}$, such that

$$\zeta\leqslant\alpha_i\leqslant\eta,\ \bar{\zeta}\leqslant\beta_i\leqslant\bar{\eta},\ \forall i\in \mathbf{Z}.$$

Lemma 3 Under the condition of theorem 2, for fixed i, we take

$$x_i^{i,1}=(x_{i,1}^{i,1},x_{i,2}^{i,1})^T=(1,\ -\alpha_i)^T,\ x_i^{i,2}=(x_{i,1}^{i,2},x_{i,2}^{i,2})^T=(1,\beta_i)^T.$$

$$x_{n+1}^{i,j}=A(m_n)x_n^{i,j},\ i,n\in \mathbf{Z},\ j=1,2,$$

then

$$x_{n,j}^{i,2}>0,\ j=1,2,\quad i,n\in \mathbf{Z}.$$

$$x_{n,1}^{i,1}>0,\ x_{n,2}^{i,1}<0,\quad i,n\in \mathbf{Z}.$$

Lemma 4 Under the condition of theorem 2, for fixed i, we obtain real numbers ε_1^i, ε_2^i from equation

$$\varepsilon_1^i x_i^{i,1}+\varepsilon_2^i A(m_i)x_i^{i,2}=b_i,$$

and define

$$x_{i+1}^i=\varepsilon_1^i x_i^{i,1},\ x_i^i=-\varepsilon_2^i x_i^{i,2}.$$

$$x_{i+n+1}^i=A(m_{i+n})x_{i+n}^i,\ x_{i-n}^i=A^{-1}(m_{i-n})x_{i-n+1}^i,\quad n=1,2,\cdots.$$

If we take

$$x_n=\sum_{i=-\infty}^{\infty}x_n^i$$

for fixed $n\in \mathbf{Z}$, then $(x_n)_{-\infty}^{\infty}$ satisfies

$$x_{n+1}=A(m_n)x_n+b_n,\ n\in \mathbf{Z}.$$

References

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[2] 贾荣庆，计算数学，4(1980)，345—349。