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Some Optimal Block-Factorial Designs*

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The theory of optimal design plays a fundamental role in experimental design. Some of the results have been widely applied to the realistic world. In[1], Kiefer proved the optimality of some block designs. In this paper some block-factorial designs are discussed.

When a block design $d_2(v_2,b,k)$ is superimposed on other block design $d_1(v_1, b,k)$, the resulting structure is called a block-factorial design and is abbreviated as d_1*d_2. The collection of all such designs is denoted by $\Omega(v_1,v_2,bk)$ *or* Ω.

If d_1,d_2 are uniform and they are orthogonal to each other, the structure can be considered as an orthogonal design with three factors. In[2] Cheng pointed out that the designs are universally optimal. If we remove the conditions of uniformity entirely or partly, optimality results of some block-factorial designs are proved.

For any $d\in\Omega(v_1,v_2,bk)$, we assume the linear model

$$EY_d=X_d\theta,\qquad VarY_d=\sigma^2 I_N \tag{1}$$

where $Y_d=(y_1,y_2,\cdots,y_N)'$ is the $N\times 1$ vector of observations, X_d is an $N\times p$ design matrix, $\theta=(\rho_1,\rho_2,\cdots\rho_{v_1},\tau_1,\tau_2,\cdots\tau_{v_2},\beta_1,\beta_2,\cdots\beta_b)'$ is a vector of unknown parameters, ρ_i is the effect of ith level of 1st factor, τ_j is the effect of jth level of 2nd factor, β_k is the effect of kth block, $\sigma^2>0$ is known or unknown and I_N is the $N\times N$ identity matrix. Then

$$X_d'X_d=\begin{pmatrix}\operatorname{diag}(r_1^{(1)},r_2^{(1)},\cdots,r_{v_1}^{(1)}) & N_{12} & N_{10}\\ N_{21} & \operatorname{diag}(r_1^{(2)},r_2^{(2)},\cdots,r_{v_2}^{(2)}) & N_{20}\\ N_{01} & N_{02} & kI_b\end{pmatrix} \tag{2}$$

where $\operatorname{diag}(a_1,a_2,\cdots,a_k)$ is the diagonal matrix with diagonal elements $a_1,a_2,\cdots,a_k$, $r_j^{(l)}$ is the number of times that the jth level of the lth factor appears in the design, N_{12} is the incidence matrix between the 1st and 2nd factors, i.e., the (s,u)th element of N_{12} is the number of times that the sth level of 1st factor and uth level of 2nd factor appear together in the design, and N_{l0} is the incidence matrix between the lth factor and the block, $l=1,2$.

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From (1), the infotmation matrix for estimating the effects of 1st factor and 2nd factor jointly is

$$\begin{pmatrix} \mathrm{diag}(r_1^{(1)},r_2^{(1)},\cdots,r_{v_1}^{(1)}) & N_{12} \\ N_{21} & \mathrm{idag}(r_1^{(2)},r_2^{(2)},\cdots,r_{v_2}^{(2)}) \end{pmatrix} - \begin{pmatrix} N_{10} \\ N_{20} \end{pmatrix} \frac{1}{k} I_b (N_{01}\ N_{02})$$

$$= \begin{pmatrix} C_{d11} & C_{d12} \\ C_{d21} & C_{d22} \end{pmatrix} \tag{3}$$

where $C_{d11} = \mathrm{diag}(r_1^{(1)},r_2^{(1)},\cdots,r_{v_1}^{(1)}) - k^{-1}N_{10}N_{01}$, $C_{d12} = N_{12} - k^{-1}N_{10}N_{02}$, $C_{d22} = \mathrm{diag}(r_1^{(2)},r_2^{(2)},\cdots,r_{v_2}^{(2)}) - k^{-1}N_{20}N_{02}$, and A^- denotes a generalized inverse of A.

If we are interested only in estimation of linear combinations $\Sigma_i c_i \rho_i$, we obtain $C_d^{(1)} = C_{d11} - C_{d12}C_{d22}^- C_{d21}$ for the infotmation matrix of d for $\rho = (\rho_1,\rho_2,\cdots,\rho_{v_1})'$. Similarly, we obtain $C_d^{(2)} = C_{d22} - C_{d21}C_{d11}^- C_{d12}$ for the information matrix of d for $\tau = (\tau_1,\tau_2,\cdots,\tau_{v_2})'$.

An optimality criterion is a function Φ: $\mathscr{B}_{v,0} \to (-\infty,\infty]$, where $\mathscr{B}_{v,0}$ is the collection of $v \times v$ nonnegative definite matrices with zero row and column sums. A design is called Φ-optimal if it minimizes $\Phi(C_d^{(1)})$ or $\Phi(C_d^{(2)})$ over the competing designs depending on which effects we are intrested in. Note that $C_d^{(1)} \in \mathscr{B}_{v_1,0}$ and $C_d^{(2)} \in \mathscr{B}_{v_2,0}$ in our setting.

Kiefer[3] introduced the notion of universal optimality. A design d^* is called universally optimal if it is Φ-optimal for all Φ satisfying (I) Φ is cônvex, (II) Φ (bC) is nonincreasing in the scalar $b \geqslant 0$, (III) Φ is invariant under any simultaneous permutation of rows and columns of C. Using a tool due to Kiefer[3], the following theorem is proved:

Theorem 1 If there is a $d^* \in \Omega$ satisfying the conditions: (I) $d^* = d_1 * d_2$, where d_1 is uniform design with parameters (v_1,b,k), d_2 is BIBD$(v_2,b,k,r^{(2)},\lambda)$; (II) $N_{12} = (v_1 v_2)^{-1}NJ$, where J is a matrix with all its entries 1, then d^* is universally optimal for the estimation of the effect of 1st factor as well as 2nd factor over Ω.

If we restrict the competing designs to a smaller class and utilize a result of Ehrenfeld[4], then some stronger optimality results can be proved. Let $\Omega^* = \Omega^*(v_1, v_2, bk) = \{d: d \in \Omega(v_1,v_2,bk),\ r_1^{(1)} = r_2^{(1)} = \cdots = r_{v_1}^{(1)} = r^{(1)}\}$, then we have

Theorem 2 Under the assumptions of theorem 1, d^* minimizes the variance of the best linear unbiased estimator of any contrast among the effect of 1st factor ρ over Ω^*.

Using Kiefer's methods, we also proved following theorems:

Theorem 3 If there is a $d^* \in \Omega$ satisfying the conditions: (I) $d^* = d_1 * d_2$, where d_1 is uniform design, d_2 is BBD(v_2,b,k), (II) $N_{12} = (v_1 v_2)^{-1}NJ$, then d^* is universally optimal for the effect of 1st factor as well as 2nd factor over Ω.

Theorem 4 If there is a $d^* \in \Omega$ satisfying the conditions: (Ⅰ) $d^* = d_1 * d_2$, where d_1 is BIBD $(v_1, b, k, r^{(1)}, \lambda^{(1)})$, d_2 is BIBD $(v_2, b, k, r^{(2)}, \lambda^{(2)})$, (Ⅱ) $N_{12} = (v_1 v_2)^{-1} NJ$, (Ⅲ) $N_{10} N_{02} = (v_1 v_2)^{-1} NkJ$, then d^* is universally optimal for the effect of 1st factor as well as 2nd factor over Ω.

References

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