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Some Properties of a Pure Jump Markov Chain after a Cut*

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Let $X=\{X_t(\omega), t\geqslant 0\}$ be a pure jump Markov chain with minimal state space $\mathrm{I}=\{0,1,2,\cdots\}$ on a probability triple (Ω,F,P). The sample function $X(\cdot,\omega)$ is right lower semi-continuous. We denote the transition matrix by $p(t)=(p_{ij}(t))$ and Q-matrix by $Q=(q_{ij})=(p'_{ij}(0))$, $i,j\in I$, where $0\leqslant\sum\limits_{j\neq i}q_{ij}=-q_{ii}=q_i<\infty$. In this paper let $q_i\neq 0$ and, without loss of generality, $p(X_0(\omega)=i)=1$. We define

$$\tau_{i0}(\omega)=0,$$

$$\tau_{in}(\omega)=inf\{t: t>\tau_{i,n-1}(\omega), X(t,\omega)\neq X(\tau_{in-1},\omega)\},$$

$$\theta_{in}(\omega)=\tau_{in+1}(\omega)-\tau_{in}(\omega),$$

$$\tau_{in}(\omega)=\tau_{i1}(\omega)+\tau_{\tau_{1n-1}}(\omega).$$

$$N_{it}(\omega)=\sup\{n: \tau_{in}(\omega)\leqslant t\} \text{ for } t\geqslant 0.$$

$$\pi_{ij}=q_{ij}/q_i \quad i\neq j, \quad \pi_{ii}=0.$$

$$W_{it}(\omega)=\tau_{iN_{it}(\omega)+1}(\omega)-t,$$

$$V_{it}(\omega)=t-\tau_{iN_{it}(\omega)}(\omega),$$

$$\theta_{iN_t(\omega)}(\omega)=\tau_{iN_{it}(\omega)+1}(\omega)-\tau_{iN_{it}(\omega)}(\omega),$$

$$G_{in}(x)=p_i(\tau_{in}(\omega)\leqslant x).$$

We have the following results:

Theorem 1 Let $F_{it}(x)=P_i(W_{it}(\omega)\leqslant x)$. Then for $x\geqslant 0$, we have

$$F_{it}(x)=e^{-q_it}(1-e^{-q_ix})+q_ie^{-q_ix}\sum_{j\neq i}\pi_{ij}\int_0^t F_{ju}(x)e^{q_iu}du \tag{1}$$

Proof If $x<0$, clearly, $F_{it}(x)=0$. If $x\geqslant 0$, then

$$F_{it}(x)=\sum_{k=0}^{\infty}P_i(\tau_{ik+1}-t\leqslant x, \tau_{ik}\leqslant t<\tau_{ik+1},$$

$$=P_i(t<\tau_{i1}\leqslant t+x)+\sum_{k=1}^{\infty}\sum_{j\neq i}P_i(X(\tau_{i1})=j)P_i(\tau_{ik}\leqslant t$$

$$<\tau_{ik+1}\leqslant t+x\,|\,X(\tau_{i1})=j).$$

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It follows from theorem 5 of [1; p. 148] that

$$G_{i_1}(x)=1-e^{-q_i x},\ x\geqslant 0,$$

and from theorem 3 of [1; p. 171] that τ_{i_1} and $\tau_{\tau_1 k-1}$ $(k>1)$ are independent (relative to P_i) with respect to $X(\tau_{i_1})=j$. Hence we obtain

$$\begin{aligned}
&P_i(\tau_{ik}\leqslant t<\tau_{ik+1}\leqslant t+x\,|\,X(\tau_{i_1})=j)\\
&=P_i(\tau_{i_1}+\tau_{\tau_1 k-1}\leqslant t<\tau_{i_1}+\tau_{\tau_1 k}\leqslant t+x\,|\,X(\tau_{i_1})=j)\\
&=\int_0^t p_i(\tau_{\tau_1 k-1}\leqslant t-u<\tau_{\tau_1 k}\leqslant t-u+x\,|\,x(\tau_{i_1})=j,\ \tau_{i_1}=u)\,dG_{i_1}(u)\\
&=\int_0^t p_j(\tau_{jk-1}\leqslant t-u<\tau_{jk}\leqslant t-u+x)\,dG_{i_1}(u)\,.
\end{aligned}$$

Thus

$$\begin{aligned}
F_{it}(x)&=e^{-q_i t}(1-e^{-q_i x})+\sum_{k=1}^{\infty}\sum_{j\neq i}\pi_{ij}\int_0^t p_j(\tau_{jk-1}\leqslant t-u<\tau_{jk}\leqslant t-u+x)\,dG_{i_1}(u)\\
&=e^{-q_i t}(1-e^{-q_i x})+\sum_{j\neq i}\pi_{ij}\int_0^t F_{jt-u}(x)\,dG_{i_1}(u)\\
&=e^{-q_i t}(1-e^{-q_i x})+q_i e^{-q_i t}\sum_{j\neq i}\pi_{ij}\int_0^t F_{ju}(x)e^{q_i u}du.
\end{aligned}$$

Theorem 2 Let $J_{it}(x)=P_i(V_{it}(\omega)\leqslant x)$. Then for $x\geqslant 0$, we have

$$J_{it}(x)=\begin{cases} q_i e^{-q_i t}\sum_{j\neq i}\pi_{ij}\int_0^x J_{ju}(x)e^{q_i u}du, & if\ \ 0\leqslant x<t\\ 1 & if\ \ x\geqslant t\ . \end{cases}\tag{2}$$

Theorem 3 Let $F_{it}(w,v)=P_i(W_{it}(\omega)\leqslant w, V_{it}(\omega)\leqslant v)$. Then for $w\geqslant 0$ and $v\geqslant 0$, we have

$$F_{it}(w,v)=\begin{cases} q_i e^{-q_i t}\sum_{j\neq i}\pi_{ij}\int_0^v F_{ju}(w,v)e^{q_i u}du, & if\ 0\leqslant w \text{ and } 0\leqslant v<t\\ e^{-q_i t}(1-e^{-q_i w})+q_i e^{-q_i t}\sum_{j\neq i}\pi_{ij}\int_0^t F_{ju}(w,v)e^{q_i u}du & \text{if } 0\leqslant w \text{ and } t\leqslant v. \end{cases}\tag{3}$$

Theorem 4 Let $H_{it}(x)=P_i(\theta_{iN_t(\omega)}(\omega)\leqslant x)$. Then for $x\geqslant 0$, we have

$$H_{it}(x)=\begin{cases} q_i e^{-q_i t}\sum_{j\neq i}\pi_{ij}\int_0^t H_{ju}(x)e^{q_i u}du & \text{if } 0\leqslant x<t\\ (e^{-q_i t}-e^{-q_i x})+q_i e^{-q_i t}\sum_{j\neq i}\pi_{ij}\int_0^t H_{ju}(x)e^{q_i u}du & \text{if } t\leqslant x\ . \end{cases}\tag{4}$$

Let I be a recurrent class. We define $a_{ii}^{(0)}(\omega)=0$,

$$\rho_i^{(1)}(\omega)=\inf\{t:t>0,X(t,\omega)\neq i\},\ \ a_{ij}^{(1)}(\omega)=\inf\{t:t>\rho_i^{(1)},X(t,\omega)=j\},$$

$$\rho_i^{(n)}(\omega)=\inf\{t:t>0,X\Big(\sum_{k=0}^{n-1}a_{ii}^{(k)}+t,\omega\Big)\neq i\},\qquad n\geqslant 1$$

$$a_{ii}^{(n)}(\omega)=\inf\{t:t>\rho_i^{(n)},X\Big(\sum_{k=0}^{n-1}a_{ii}^{(k)}+t,\omega\Big)=j\},\qquad n\geqslant 1$$

$$S_{in}(\omega)=\sum_{k=0}^{n}\alpha_{ii}^{(k)}(\omega),\quad \tilde{N}_{it}(\omega)=\sup\{n; S_{in}\leqslant t\},\quad \tilde{W}_{it}(\omega)=S_{i\tilde{N}_{it}(\omega)+1}(\omega)-t,$$

$$\tilde{V}_{it}(\omega)=t-S_{i\tilde{N}_{it}(\omega)}(\omega),\quad \tilde{\theta}_{i\tilde{N}_{t}(\omega)}(\omega)=S_{i\tilde{N}_{it}(\omega)+1}(\omega)-S_{i\tilde{N}_{it}(\omega)}(\omega),$$

$$R_i(x)=P_i(\alpha_{ii}^{(1)}(\omega)\leqslant x).$$

Theorem 5 Let $\tilde{F}_{it}(x)=P_i(\tilde{W}_{it}(\omega)\leqslant x)$. Then for $x\geqslant 0$,

$$\tilde{F}_{it}(x)=R_i(x+t)-R_i(t)+\int_0^t\tilde{F}_{iu}(x)R_i'(t-u)du. \tag{5}$$

Theorem 6 Let $\tilde{J}_{it}(x)=P_i(\tilde{V}_{it}(\omega)\leqslant x)$. Then for $x\geqslant 0$,

$$\tilde{J}_{it}(x)=\begin{cases}\int_0^x\tilde{J}_{iu}(x)R_i'(t-u)du & \text{if } 0\leqslant x<t\\ 1 & \text{if } x\geqslant t\ ,\end{cases} \tag{6}$$

Theorem 7 Let $\tilde{F}_{it}(w,v)=P_i(\tilde{W}_{it}(\omega)\leqslant w,\tilde{V}_{it}(\omega)\leqslant v)$. Then for $w\geqslant 0$ and $v\geqslant 0$.

$$\tilde{F}_{it}(w,v)=\begin{cases}\int_0^v\tilde{F}_{iu}(w,v)R_i'(t-u)du, & \text{if } 0\leqslant w \text{ and } 0\leqslant v<t\\ R_i(t+w)-R_i(t)+\int_0^t\tilde{F}_{iu}(w,v)R_i(t-u)du, & \text{if } 0\leqslant w \text{ and } v\geqslant t.\end{cases} \tag{7}$$

Theorem 8 Let $\tilde{H}_{it}(x)=P_i(\tilde{\theta}_{iN_{it}(\omega)}(\omega)\leqslant x)$. Then for $x\geqslant 0$

$$\tilde{H}_{it}(x)=\begin{cases}\int_0^t\tilde{H}_{iu}(x)R_i'(t-u)du & \text{if } 0\leqslant x<t\\ R_i(x)-R_i(t)+\int_0^t\tilde{H}_{iu}(x)R_i'(t-u)du & \text{if } x\geqslant t\ .\end{cases} \tag{8}$$

Proof of theorems 2 through 8 and other results will appear elsewhere;

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