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ON BEST LOCAL APPROXIMATION

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The purpose of this paper is to introduce certain aspects concerning the problem of best local approximations. Especially in multipoint best local approximation, there are many intuitively quite clear yet unsolved interesting problems. We will discuss some recent results and some unsolved problems.

1. Introduction

The problem of best local approximation was introduced and studied in a paper by Chui, Smith and Shisha [5] in association with the study of Padé approximants. The initiation of this could be dated back to results of J. L. Walsh [13, 14] who found the Padé approximants of an analytic function over a domain (real or complex) can be obtained as a "limit" of the net of the best rational approximants by shrinking the domain to a single point. A special case of this, when approximating functions are chosen to be polynomials, the limit is the Taylor polynomial of the same degree which was already known to Walsh in 1934 [12]. These results mentioned above were concerning the problem over a single domain (real or complex) and study the limiting behavior of certain approximating functions as the domain "shrinking" to a single point. In 1979, Su [10] studied the problem, which was suggested by Chui, over two disjoint intervals. This study sets up the initial steps for multipoint best local approximation. Later, Beatson and Chui [2] formally introduced the multipoint problem and obtained some promising results. We shall introduce these results in a section after we define Padé approximants as well as best local approximations and cite a few important results. We end with a section introducing a few unsolved problems.

2. Padé approximants and best local approximations

The subject of the Padé approximants is very old [9]. There are several equivalent definitions; people who are interested in this should consult the works by

Padé [9], Wall [11], Baker [1] and Chui [3]. For our purpose, we shall start with the following:

Definition 2.1: Consider a formal power series with complex (or real) coefficients c_k

$$f(z)=\sum_{k=0}^{\infty}c_kz^k. \tag{2.1}$$

The $[m/n]$ Padé approximant of f (at the origin) is the unique function $[m/n]:=[m/n]f:=p_m/q_n$, where $q_n\neq 0$ and

$$p_m(z)=\sum_{k=0}^{m}a_kz^k \text{ and } q_n(z)=\sum_{k=0}^{n}b_kz^k,$$

such that

$$f(z)q_n(z)-p_m(z)=dz^{m+n+1}+h.o.t.$$

The following theorem collects a few equivalent formations;

Theorem 2.1 *Let f be a formal power series (as in eq. 2.1) and let f_{m+n} be the $(m+n)^{th}$ partial sum of f. The following are equivalent to the definition of Padé approximants $[m/n]f=p_m/q_n$ of f:*

(i) *$f(z)q_n(z)-p_m(z)=sz^u+h.o.t.$, where $u\leqslant\infty$ is as large as possible and $q_n\neq 0$;*

(ii) *$f(z)-p_m(z)/q_n(z)=tz^v+h.o.t.$, where $v\leqslant\infty$ is as large as possible and $q_n\neq 0$;*

(iii) *$(f_{m+n}q_n-p_m)^{(j)}(0)=0$ for $j=0,\ldots,m+n$ and $q_n\neq 0$. Here the formal derivatives are taken at $z=0$.*

The following important result is due to Walsh [13].

Theorem 2.2 *Let $f(z)$ be an analytic function at $z=0$ and have a power series expansion as in eq. (2.1). Let ε be sufficiently small, m and n are fixed nonnegative integers; Let $r_\varepsilon=r_\varepsilon(m,n,z)$ denote the best rational approximation of type (m,n) w.r.t. L_p norm on a disk δ: $|z|\leqslant\varepsilon$. Suppose*

$$\Delta_{m-1,n-1}=\begin{vmatrix} c_m & c_{m-1} & \cdots & c_{m-n+1}\\ c_{m+1} & c_m & \cdots & c_{m-n+2}\\ \cdots & \cdots & \cdots & \cdots\\ c_{m+n-1} & c_{m+n} & \cdots & c_m\end{vmatrix}\neq 0; \tag{2.2}$$

Then, as $\varepsilon\to 0^+$, $r_\varepsilon\to[m/n]f$ on any compact set containing no zeros of $q_n(z)$.

Later, in 1974 Walsh [14] obtained a similar result for $f\in C^{m+n+1}[0,1]$, with the same normality conditions as in eq.(2.2) and the norm used in this is L_∞.Chui, Shisha and Smith [4] obtained the same result without assuming (2.2). In 1975, Chui, Shisha, and Smith [5] developed the idea of best local approximation which is defined as follows:

Let G be a class of real-valued functions in $C[0,\delta]$ with $\delta>0$. Suppose that for each ε, $0<\varepsilon\leqslant\delta$, a function $f\in C[0,\delta]$ has the best uniform approximant $p_\varepsilon(f)$ on $[0,\varepsilon]$ from G, i.e.,

$$\|p_\varepsilon(f)-f\|_{[0,\varepsilon]}=\inf\{\|p-f\|_{[0,\varepsilon]}:p\in G\}$$

where $\|\cdot\|_{[0,\varepsilon]}$ is the supremum norm over $[0,\varepsilon]$. If as $\varepsilon\to 0^+$, $p_\varepsilon(f)$ converges to some function $p_0(f)$ uniformly on some interval $[0,\varepsilon_0]$, $\varepsilon_0>0$, then call $p_0(f)$ the best local approximant of f. If $G:=\text{span}\{u_1,\ldots,u_n\}$, $u_i\in C^n[0,\varepsilon]$ is a Haar system, then the best L_p approximation to $f\in[0,\varepsilon]$ exists uniquely on any compact set, hence it is natural to study the best local approximation over a Haar system. A key result in the L_∞ setting is included in the following theorem established by Chui, Shisha and Smith [5].

Theorem 2.3 *The net* $\{p_\varepsilon(f)\}$, $0<\varepsilon\leqslant\delta$ *converges as* $\varepsilon\to 0^+$, *w.r.t.* L_∞ *norm, for every* $f\in C^n[0,\delta]$ *if and only if the* $n\times n$ *matrix*

$$A_n:=[u_j^{(i-1)}(0)]:=\begin{pmatrix} u_1(0) & \cdots\ u_n(0)\\ u_1'(0) & \cdots\ u_n'(0)\\ \cdots & \cdots\ \cdots\\ u_1^{(n-1)}(0) & \cdots u_n^{(n-1)}(0)\end{pmatrix}$$

is nonsingular; Furthermore, in case of convergence for a given $f\in C^n[0,\delta]$, the limit $p_0(f)$ *is* in G *and satisfies*

$$p^{(j)}(f)(0)=f^{(j)}(0),\quad j=0,\ldots,n-1.$$

As an application of this, they also obtained the Padé approximant as a limit of the so called best quasi rational approximation. In the L_2 setting with G a smooth Haar system, similar results can be seen in [6]. When G is an algebraic polynomial and with the L_p norm the result is done in Su [10], where the two-point local approximation was also studied. we shall introduce this in the next section.

3. Multipoint best local approximation

Let f be a sufficiently smooth function on a union of m nondegenerate disjoint intervals I_ε, and g_ε be a best approximant of f from a certain class of smooth functions G with respect to a certain norm (e.g., L_p norm). Suppose, as $\varepsilon\to 0^+$, I_ε shrinks to a union of m disjoint points and g_ε converges to $g_0\in G$. Then we say g_0 is an m—point best local approximant of f over I_ε with respect to the specified norm. The natural questions are the existence, uniqueness and characterization of m—point best local approximation.

The following theorem generalizes the one point L_p result in Su [10] to m—point where $m\geqslant 2$ by Beatson and Chui [2]. Let

$$\pi_{mN}=\{p:\ p \text{ is algebraic polynomial of } \deg\leqslant mN-1\}.$$

Theorem 3.1 *Let $1\leqslant p<\infty$ and $f\in C^{N-1}(\delta)$ where $N\geqslant 1$. For each ε, $0<\varepsilon\leqslant\delta$, let p_ε be the best $L_p(I_\varepsilon)$ approximant to f from π_{mN}. Then the net $\{p_\varepsilon\}$ converges coefficientwise, as $\varepsilon\to 0^+$, to some $p_0\in\pi_{mN}$. Furthermore P_0 is the unique polynomial in π_{mN} which satisfies the interpolation conditions*

$$p_0^{(j)}(x_i)=f^{(j)}(x_i),\qquad i=1,2,\dots,m;\qquad j=0,1,\dots N-1.$$

Clearly, in this case the number of parameters match with the number of "data" at x_i, $i=1,\dots,m$, and the best local approximant turned out to be the Hermite interpolation polynomials which enjoys the oscillating interpolating property evenly over the m points. What happens if the number of parameters does not match the number of data? Only a few special cases are known. The following result is in Su [10].

Theorem 3.2 *Let $f\in C^n[I_\varepsilon]$, $I_\varepsilon=[-1,-1+\varepsilon]\cup[-1,1-\varepsilon]$, ε is sufficiently small. Then the two-point best L_2 local approximant P_0 of f from π_k exists and is uniquely determined by the following conditions:*

(i) *If $k=2n$, then*

$$(p_0-f)^{(j)}(\pm 1)=0\qquad j=0,1,\dots,n-1.$$

(ii) *If $k=2n+1$, then*

$$(p_0-f)^{(j)}(\pm 1)=0,\qquad j=0,1,\dots,n-1$$

and

$$p_0^{(n)}(1)+(-1)^n p_0^n(-1)=f^{(n)}(1)+(-1)^n f^{(n)}(-1).$$

Let $G_f:=\{p:p\in\pi_{2k+1},(p-f)^{(j)}(\pm 1)=0,\ j=0,1,\dots,n-1\}$. Then part (ii) above is equivalent to say p_0 is the solution of the (discrete) l_p norm problem, where $p=2$

$$\|p_0^{(n)}-f^{(n)}\|_{l_p}=\inf_{p\in G_f}\|(p-f)^{(n)}\|_{l_p}$$

where

$$\|g\|_{l_p}:=\{(g(1))^p+(g(-1))^p\}^{1/p},\qquad p=2$$

On replacing L_2 and l_2 by L_∞ and l_∞ respectively, where $\|g\|_{L_\infty}=\max\{|g(1)|,|g(-1)|\}$, the same statement is true, but the proof is quite complicated and was given in [2]. If the number of points $m\geqslant 3$, let x be a set of m distinct points, and

$I_\varepsilon=\bigcup_{i=1}^{m}[x_i-\varepsilon,x_i+\varepsilon]$ with ε sufficiently small. Furthermore let $f\in C^1(I_\varepsilon)$ and

$$H_f:=\{p\in\pi_{2m-1}:(p-f)(x_i)=0,\ i=1,2,\dots,m\}.$$

Then the following result is included in [2].

Theorem 3.3 *Let $f\in C'(I_\varepsilon)$. For each ε, $0<\varepsilon\leqslant\delta$, let $p_\varepsilon(f)$ be the best uniform approximation of f on I_ε from π_{2m-1}. Then the net $\{p_\varepsilon(f)\}$ converges coefficientwise, as $\varepsilon\to 0^+$, to some $p_0\in\pi_{2m-1}$. Furthermore, p_0 is the unique polynomial in H_f which minimizes max $\{|(p-f)'(x_i)|:i=1,\dots,m\}$ over all $p\in H_f$.*

In all multipoint cases stated above the burden of proof seems to lie upon the proof of convergence of p_ε as $\varepsilon \to 0^+$. Except in the L_2 case, using the linear structure and the symmetry as was done in Su [10] the convergence result is readily extended to the case over $f \in C^{(n)}(I_\varepsilon)$, with arbitrary integer $n \geqslant 0$, and x can be any set of even number of equally spaced points. So far, much need be done in multipoint best local approximations. From the above known results, it seems reasonable to conjecture that if the best multipoint L_p approximation exists it has to interpolate evenly over all the points as 'high' an order as possible and minimize the l_p over the remaining data at the discrete points. But no proof is known besides those cases we have cited. We shall state a few unsolved problems in the next section.

4. Unsolved problems

(i) Consider the two-point problem as in Theorem 3.2 over algebraic polynomials but in the L_p setting. Do we have a similar result as in Theorem 3.2? Here $1 \leqslant p \leqslant \infty$.

(ii) Cosider the two-point problem as in Theorem 3.2 but over a general Haar system and using the L_p norm. Do we have some result as a generalization of Theorem 2.3?

(iii) m—point problem, same questions as in (i) and (ii).

(iv) m—point problem but G is rational functions, any norm, do we get multipoint Padé approximant as a limit?

(v) If the disjoint intervals have different "sizes" or the intervals converge to zero at a different rate, same questions as (i) – (iv).

(vi) If the disjoint intervals are replaced to disjoint complex domains and f is an analytic function, same questions as in (i) – (v).

Finally we note that the best approximation on small intervals has been studied by many authors. We cite only Maehly and Witzgall [7] and Meinardus [8].

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