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On a Theorem of Szegö*

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Let S be the class of functions $f(z)=z+a_2z^2+\cdots$, regular and schlicht in $|z|<1$ and $S_n(z)=z+a_2z^2+\cdots+a_nz^n$. Szegö[1] proved that $S_n(z)$ are schlicht in $|z|=\rho<\frac{1}{4}$ for $n=2,3\cdots$. The radius $\rho_0=\frac{1}{4}$ is best possible. The author Hu Ke predicates that ρ_0 is the radius ρ_* of starlikeness of $S_n(z)$. Here we prove that the assertion is true, i. e. we have the following theorem.

Theorem Let $f(z)=z+a_2z^2+\cdots$. Then the functions $S_n(z)$ is starlike with respect to 0 in $|z|=\rho<\frac{1}{4}$.

Particularly, $f\in S^*$ was obtained by Wu Zhuo Ren[2]. The case $n=3$ of the following proof is established by Hu, the others by Pan.

Lemma 1 Let $f\in S$. Then

$$\operatorname{Re} z\frac{f'(z)}{f(z)}\geqslant\frac{r^*-r}{r^*+r},\quad \text{for } r<r^*=\tanh\frac{\pi}{4}.$$

Proof Since the radius of starlikeness of S is r^*, it follows that $g(z)=\frac{f(r^*z)}{r^*}\in S$, i. e.

$$\operatorname{Re} z\frac{g'(z)}{g(z)}\geqslant\frac{1-r}{1+r}$$

However,

$$\operatorname{Re} z\frac{g'(z)}{g(z)}=\operatorname{Re} zr^*\frac{f'(r^*z)}{f(r^*z)}\geqslant\frac{1-r}{1+r},$$

therefore we have

$$\operatorname{Re} z\frac{f'(z)}{f(z)}\geqslant\frac{1-\frac{r}{r^*}}{1+\frac{r}{r^*}}=\frac{r^*-r}{r^*+r},\quad \textit{for } r\leqslant r^*.$$

Lemma 2 Let $f\in S$ and $R_n(z)=\sum_{k=n+1}^{\infty}a_kz^k$. Then

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$$|R_n(z)|\leqslant 1.07\,\frac{r^{n+1}(n+1-nr)}{(1-r)^2}$$

$$|R_n'(z)|\leqslant 1.07\,r^n\,\frac{(n+1)^2\,(1-r)^2+2(n+1)r(1-r)+r(1+r)}{(1-r)^3}.$$

Making use of the fact that $|a_n|\leqslant 1.07n$ for $f\in S(n=2,3\cdots)$[3], the lemma can be verified directly.

The proof of Theorem: Our proof proceeds in a number of stages.

(a) The case $n\geqslant 4$. Since $f(z)=S_n(z)+R_n(z)$, we have

$$\begin{aligned} I&=\operatorname{Re} z\,\frac{S_n'(z)}{S_n(z)}=\operatorname{Re} z\,\frac{f'(z)-R_n'(z)}{f(z)-R_n(z)}\\ &=\operatorname{Re} z\,\frac{f'(z)}{f(z)}+\operatorname{Re}\left\{z\,\frac{f'(z)R_n(z)-R_n'(z)f(z)}{f(z)(f(z)-R_n(z))}\right\}\\ &\geqslant \operatorname{Re}\frac{zf'(z)}{f(z)}-\frac{\left|z\dfrac{f'(z)}{f(z)}\right||R_n(z)|+|z|\,|R_n'(z)|}{\big||f(z)|-|R_n(z)|\big|} \end{aligned}\tag{1}$$

It is sufficient to prove $I\geqslant 0$ for $|z|=\frac{1}{4}$. By Lemma 2, it is trivial that ($l_0=1.07$)

$$|R_n(z)|\leqslant l_0\,\frac{r^{n+1}(n+1-nr)}{(1-r)^2}\leqslant l_0\frac{r^5(5-4r)}{(1-r)^2},$$

and $|f(z)|>|R_n(z)|$, *a fortiori*,

$$\big||f(z)|-|R_n(z)|\big|=|f(z)|-|R_n(z)|\geqslant\frac{r}{(1+r)^2}-l_0\,\frac{r^5(5-4r)}{(1-r)^2}.$$

In virtue of Lemma 1 and (1), we get

$$\begin{aligned} I\geqslant&\frac{r^*-r}{r^*+r}-l_0\,\frac{r^4(1+r)^2}{(1-r)^2-r^4(5+4r)l_0(1+r)^2}\left[\frac{1+r}{1-r}(5-4r)\right.\\ &\left.+\frac{5^2(1-r)^2+10r(1-r)+r(1+r)}{1-r}\right]=I_1-I_2,\ \text{say}. \end{aligned}$$

Substituting $r=\frac{1}{4}$ into I_1 and I_2, the calculations show $I_1=0.4475$ and $I_2=0.3224$, i. e. $I>0$.

(b) The case $n=2$. The proof of the case is very simple, for

$$\begin{aligned}\operatorname{Re}(1+2a_2z)(1+\overline{a_2}\,\overline{z})&=1+\frac{1}{8}|a_2|^2+\frac{3}{4}\operatorname{Re}\left[a_2e^{i\theta}\right]\\ &\geqslant\frac{1}{8}(4-|a_2|)(2-|a_2|)\geqslant 0.\end{aligned}$$

(c) The case $n=3$. Let us denote $a_2=-x-iy$ and $a_3=s+it$, it is sufficient to prove $F(x,\ y,\ s,\ t)>0$, where

$$F(x,\ y,\ s,\ t)=256+32(x^2+y^2)+3(s^2+t^2)-192x-20(sx+yt)+64s$$

Since $F(x, y, s, t) \geqslant F(x, |y|, s, |t|)$, without loss of generality, we assume that $y \geqslant 0$ and $t \geqslant 0$, and we note the following known inequalities:

$$x^2+y^2 \leqslant 4, \quad s^2+t^2 \leqslant 9 \quad and \quad (x^2-y^2-s)^2+(t-2xy)^2 \leqslant 1. \tag{2}$$

Then the proof proceeds in a number of stages.

(i) If $x < \frac{2}{3}$, then $F(x, y, s, t) > 0$. Since $|s| \leqslant 3$, we have

$$F = 256+32x^2-192x+3s^2+(64-20x)s-\frac{4}{3}y^2+3\left(t-\frac{10}{3}y\right)^2$$

$$\geqslant 91-132x-\frac{16}{3}+32x^2 > 10. \tag{3}$$

(ii) If $s > 0$, then $F > 0$. In virtue of (2) and (3),

$$F > 256+32x^2-192x-\frac{4}{3}y^2 \geqslant 256+32x^2-192x-\frac{4}{3}(4-x^2)=\varphi(x), \text{ say.}$$

Since $\varphi(x)$ is decreasing in x, we have $F(x) \geqslant \varphi(2)=0$. Noting that it is impossible that $s=y=0$, when $x=2$, by the last inequality in (2), it follows $F(x) > 0$.

(iii) Only the case $x \geqslant \frac{2}{3}$, $y > 0$, $s < 0$, $t > 0$ remains to be proved. Let us denote $w=|s|$ and $F(x, y, -s, t)=F_1(x, y, w, t)$ where

$$F_1(x, y, w, t)=256+32(x^2+y^2)-192x+3(w^2+t^2)-(64-20x)w-20yt.$$

For $w > 0$, the last inequality of (2) becomes $x^2 \leqslant 1+y^2-w$. Fixed y and w, then $\varphi_1(x)=32x^2-(192-20w)x$ is decreasing for $x \in [0, \sqrt{1+y^2-w}]$. It follows that

$$F_1(x, y, w, t) \geqslant 256+3t^2-20yt+3w^2+32y^2-64w+32(1+y^2-w)$$

$$-(192-20w)\sqrt{1+y^2-w}. \tag{4}$$

We again consider the following two cases.

(A) If $y \geqslant 0.9$, then $3t^2-20ty > 27-60y$. From (4), and

$$(192-20w)\sqrt{1+y^2-w} \leqslant \frac{1}{2}\frac{\sqrt{5}}{192}(192-20w)^2+\frac{96}{\sqrt{5}}(1+y^2-w),$$

we have

$$F_1 \geqslant 288+64y^2-20ty+3t^2+3w^2-96w-\frac{\sqrt{5}}{384}(192-20w)^2-\frac{96}{\sqrt{5}}(1+y^2-w)$$

$$=30.4+3t^2+21y^2-20yt+0.67w^2-8.35w=\varphi_3(y, w, t).$$

First, setting $w \geqslant 2$ then $t \leqslant \sqrt{9-w^2} \leqslant \sqrt{5}$, we obtain that

$$\varphi_3(y, w, t) > \varphi_3(y, 3, \sqrt{5}) > 26.2-20\sqrt{5}y+21y^2=\varphi_4(y)$$

$$> \varphi_4(1.06479) > 2.39.$$

Secondly, we consider $w \in (0, 2]$. The $y \geqslant 0.9$ gives $3t^2-20ty > 27-60y$. Hence

$$\varphi_3(y, w, t) > \varphi_3(y, 2, 3) > 43-60y+21y^2=\varphi_5(y) \geqslant \varphi_5(1.42857) > 1.428.$$

(B) If $y < 0.9$, the inequality $x^2+w \leqslant 1+y^2$ gives $x \leqslant 1.17$ and $w \leqslant 1.36555$, because $x \geqslant \frac{2}{3}$.

$$F_2(x,\ y,\ w,\ t)\geqslant 256+32x^2-\frac{4}{3}y^2-(64-20x)w+3w^2-192x=F_2(x,\ y,\ w)$$
$$\geqslant F_2(x,\ 0.9,\ 1.36555)>174+32x^2-165x-\frac{4}{3}y^2$$
$$>174+43-193=24.$$

The theorem follows at once.

References

[1] Szegö. G., Zur theorie der Schlichten Abbildungen, *Math. Annalen*, 100. (1928), 188—211.

[2] Wu Zhuoren, Shuxue Xuebao, V. 6, No. 3 (1956), 476—481.

[3] Horowitz, D. A., Further Refinement for Coefficient Estimates of Univerlent Functions, *Proc. A. M. S.* 71 (1978), 217—222.

论 Szegö 的定理

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设$f(z)=z+a_2z^2+\cdots\in S$。Szegö 证明：$S_n(z)=z+a_2z^2+\cdots+a_nz^n$（$n=2$，$3\cdots$）在$|z|<\frac{1}{4}$内单叶。$\rho_0=\frac{1}{4}$最好的，我们证明了更强的结果：

定理：若$f(z)\in S$，则$S_n(z)$（$n=2$，$3\cdots$）在$|z|<\frac{1}{4}$内关于原点成星形。

当$f\in S^*$时为吴卓人所得。