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Some Notes on Submanifolds of an Euclidean Space with Conformal Gauss Map*

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Let (M,g) be an m-dimensional Riemannian manifold and $i:M\to E^n$ an isometric immersion of (M,g) into an n-dimensional Euclidean space E^n. Let $V\subset M$ be an open set in which the immersion $i:M\to E^n$ is given by $x^h=x^h(y^\alpha)$, $(h=1,\cdots,n;\ \alpha=1,\cdots,m)$. Here and in the sequel $x^h(h=1,\cdots,n)$ are rectangular coordinates of E^n and $y^\alpha(\alpha=1,\cdots,m)$ are local coordinates of a generic point in V. The tangent plane $iM_{ip}=i(M_p)$, $p\in V$, of iM can be considered after a suitable parallel displacement as a point $\Gamma_i(p)$ of the Grassmann manifold $G(m,n\text{-}m)$. The mapping $\Gamma:iM\to G(m,n\text{-}m)$, $ip\longmapsto\Gamma(ip)=\Gamma_i(p)$ is called the Gauss map. The mapping $\Gamma_i:M\to G(m,n\text{-}m)$, $p\longmapsto\Gamma_i(p)$ is called the Gauss map associated with the immersion i, and $\Gamma_i(M)=\Gamma(iM)$ the Gauss image of M.[1]

Assume M be a connected C^∞ manifold and the map Γ_i be regular. Let g_i be the Riemannian metric induced from E^n on iM and G_i be the Riemannian metric induced from the standard Riemannian metric $\tilde{g}$ of $G(m,n\text{-}m)$ on $\Gamma_i(M)$. If G_i equals $e^{2\rho}g_i$ for some C^∞ function ρ, namely, $\Gamma:(iM,g_i)\to(\Gamma(iM),G_i)$ is a conformal mapping, then the Gauss map Γ_i is said to be conformal. Particularly, if ρ is constant, Γ_i is said to be homothetic. Y. Muto [2] studied submanifolds in E^n with homothetic Gauss map. In this paper we study submanifolds in E^n with conformal Gauss map and obtain the following results.

Theorem 1 Let $i:(M,g)\to E^n$ be an isometric immersion of an m-dimensional locally indecomposable Riemannian manifold (M,g) in E^n. Assume that the Gauss map $\Gamma_i:M\to G(m,n\text{-}m)$ is regular. Let G_i be the Riemannian metric induced on $\Gamma_i(M)$. Then.

(1) $\Gamma_i:M\to G(m,n\text{-}m)$ is conformal if and only if G_i is recurrent in (M,g).

(2) $\Gamma_i:M\to G(m,n\text{-}m)$ is homothetic if and only if G_i is parallel in (M,g).

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Theorem 2 Let $i:(M,g)\to E^n$ be an isometric immersion of an m-dimensional Riemannian manifold (M,g) in E^n $(n>m\geqslant 3)$. Assume that the Gauss map $\Gamma_i:M\to G(m,n\text{-}m)$ is conformal, and the Gauss image $\Gamma_i(M)$ is totally umbilical and not totally geodesic. Then immersion i is pseudo-umbilical if and only if (M,g) has the constant sectional curvature.

Remark If $\Gamma_i(M)$ is totally geodesic, then it is locally symmetric. Assume that Γ_i is homothetic. Then (M,g) is also locally symmetric. (see [2] Theorem 3.2)

Theorem 3 Let $i:(M,g)\to E^n$ be as above. Assume that $\Gamma_i:M\to G(m,n\text{-}m)$ is conformal. If i is minimal, then Γ_i is homothetic, namely $G_i=c^2g_i$. If moreover $\Gamma_i(M)$ is totally umbilical and not totally geodesic, then (M,g) has the negative constant sectional curvature $K=-\dfrac{c^2}{m-1}$. Conversely, if (M,g) has the negative constant sectional curvature K and $G_i=-(m-1)Kg_i$, then the immersion i is minimal.

The outline of the proofs:

Let $g_{\mu\lambda}$ be the components of the metric g_i of iM on V, $H^h_{\mu\lambda}(\mu,\lambda=1,\cdots,m;\ h=1,\cdots,n)$ the components of the second fundamental form of iM, and $G_{\mu\lambda}(\mu,\lambda=1,\cdots,m)$ the components of the metric G_i of $\Gamma_i(M)$. We have

$$G_{\mu\lambda}=\Sigma_h g^{\alpha\beta}H^h_{\mu\alpha}H^h_{\lambda\beta},\qquad (1)$$

where $g^{\alpha\beta}(\alpha,\beta=1,\cdots,m)$ is the contravariant components of g_i.

Assume that the metric G_i of $\Gamma_i(M)$ is recurrent in (M,g), namely, in each V,

$$\nabla_\alpha G_{\mu\lambda}=a_\alpha G_{\mu\lambda},\qquad (2)$$

where ∇ represents covariant differentiation in (M,g) and $a_\alpha(\alpha=1,\cdots,m)$ are the components of some vector field on M. If (M,g) is locally indecomposable, then

$$G_{\mu\lambda}=\rho g_{\mu\lambda}\qquad (3)$$

for some function ρ, namely, $(\Gamma_i(M),G_i)$ is conformal to (M,g). Hence, $\Gamma_i:(M,g)\to(\Gamma_i(M),G_i)$ is conformal. If G_i is parallel in (M,g), namely, in each V,

$$\nabla_\alpha G_{\mu\lambda}=0,\qquad (4)$$

then we have (3) with $\rho=$const., Hence, Γ_i is homothetic. The converses are obvious.

It is shown that the totally umbilical submanifold of a locally symmetric Riemannian manifold is conformally flat if it is not totally geodesic. $G(m,n\text{-}m)$ is locally symmetric. Therefore, if $\Gamma_i(M)$ is totally umbilical and not totally geodesic, then $\Gamma_i(M)$ is conformally flat.

Assume that iM is pseudo-umbilical, namely,

$$\Sigma_h H^h_{\mu\lambda}H^h=\theta g_{\mu\lambda},\qquad (5)$$

for some function θ, where $H^h = \frac{1}{m} g^{\mu\lambda} H^h_{\mu\lambda}$ are the components of the mean curvature vector of iM. Using the Gauss equation, we can show that (M, g) is an Einstein manifold. Because Γ_i is conformal, (M, g) is conformally flat. Hence, (M, g) has the constants sectional curvature.

Conversely, if (M, g) has the constant sectional curvature K, then from the Gauss equation and $G_{\mu\lambda} = e^{2\rho} g_{\mu\lambda}$ we can obtain (5) which proves that $i:(M, g) \to E^n$ is pseudo-umbilical.

The immersion $i:(M, g) \to E^n$ is minimal if and only if $H^h = 0$, $(h = 1, \cdots, n)$. Then we can prove the theorem 3.

References

[1] Muto Y., J. *Math. Soc. Japan*, 32(1978). 85—100.

[2] Muto Y., J. *Math. Soc. Japan*, 32(1980). 531—555.