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On Coflat Modules and the Weak Global Dimension of Rings*

Chen Jianai

(Beijing Teacher's College)

We continue the discussion on coflat modules as defined in [2]. Further properties of coflat modules are given, especially the relations between coflat resolutions and the functor Ext. Weak global codimension of rings will then be introduced and it's agreement with weak global dimension shown for coherent rings, Since the later is left-right symmetrical, this offers then an alternative to demonstrate the fact that left and right global dimension are the same for Notherian rings over which each coflat module is injective.

All rings considered are associative with an identity element and all (right) modules are left (right) unitary R-modules. $_R\underline{M}$ means the category of R-modules.

We recall that a module I is coflat iff for each finitely generated submodule N of R^n $(n\in\mathbb{N})$ every f is extendable. From this condition it follows plainly that each coflat module over a Noetherian ring is injective.

$$\begin{tikzcd} O \arrow[r] & N \arrow[r] \arrow[d, "f"'] & R^n \arrow[dl, dashed] \\ O \arrow[r] & I & \end{tikzcd}$$

Lemma 1 Let $O\to E'\to E\to E''\to O$ be an $_R\underline{M}$- exact sequence.

(1) If E' is f. g. (=finitely generated) and E is f. p. (=finitely presented), then E'' is f.p..

(2) If E is f. g. and E'' is f.p., then E' is f.g..

(3) If both E' and E'' are f.p., then E is f.p..

Proof This is straightforward and may be omitted.

Let $E\in$ f.p. stands for E being a f. p. module and $\mathrm{Ext}^k(-,I)_{\mathrm{f.p.}}=0$ for the phrase "$\mathrm{Ext}^k(-,I)=0$ for every f. p. modules".

Proposition 1 Let I be any module. Then I is coflat iff $\mathrm{Ext}^1(-,I)_{\mathrm{f.p.}}=0$.

Proof If I is coflat, each exact sequence $O\to I\xrightarrow{\varepsilon}X\xrightarrow{\delta}E\to O$ is pure exact for any $E\in$ f.p.. It follows that E is flat and therefore projective. So $\mathrm{Hom}_R(E,-)$ is an exact functor and $\mathrm{Ext}^1(E,I)=\ker\mathrm{Hom}_R(E,\delta)/\mathrm{im}\ \mathrm{Hom}_R(E,\varepsilon)=0$.

Conversely, let I be any module. Consider the diagram

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$$\begin{array}{ccccccccc} O & \to & N & \longrightarrow & R^n & \longrightarrow & R^n/N & \to & O \\ & & f\downarrow & & \downarrow & & \downarrow g & & \\ O & \to & I & \to & I \amalg_N R^n & \to & R^n/N & \to & O \end{array}$$

where N is any f.g. submodule of R^n and $I \amalg_N R^n$ denotes the pushout of the left square. The sequences are both exact. By lemma 1 R^n/N is f.p. and therefore projective because of $\mathrm{Ext}^1(-,I)_{\mathrm{f.p.}}=0$. g is liftable, or equivalently (by homotopy lemma), f is extendable. This shows the coflatness of I.

Definition 1 Let B be a module. Then the coflat dimension of B $\mathrm{cfd}(B)\leqslant n$ if there exists a coflat resolution $O\to B\to I^0\to I^1\to\cdots\to I^n\to O$.

If no such finite resolution exists, define $\mathrm{cfd}(B)=\infty$. If n is the least of such integer, define $\mathrm{cfd}(B)=n$.

Definition 2 Let $O\to B\xrightarrow{\varepsilon} I^0\xrightarrow{d^0} I^1\xrightarrow{d^1}\cdots$ be a coflat resolution. Denote $X^0=\mathrm{im}\,\varepsilon$ and $X^n=\mathrm{im}\,d^{n-1}$, $n\geqslant 0$. X^n is called the n-th coyoke of B.

Definition 3 Let R be a ring. Its left weak codimension is defined by

$$\mathrm{lwc}D(R)=\sup\{\mathrm{cfd}(B)\mid B\in {}_R\underline{\mathbf{M}}\}.$$

The problem of left-right symmetry suggests a study on coherent rings.

Lemma 2 Let R be a left coherent ring. Then every f.g. submodule of R^n ($n\geqslant 1$) is f.p..

Proof For $n=1$ the conclusion is just the definition of coherence. Assume the result for $n-1$ and consider the diagram

$$\begin{array}{ccccccccc} O & \to & N\cap R^{n-1} & \to & N & \xrightarrow{p} & P(N) & \to & O \\ & & \downarrow & & \downarrow & & \downarrow & & \\ O & \longrightarrow & R^{n-1} & \longrightarrow & R^n & \xrightarrow{p} & R & \to & O \end{array}$$

where p is the n-th projection from R^n to R. $p(N)$ is a f.g. ideal of R. It is also f.p. because R is coherent. By lemma 1 $N\cap R^{n-1}$ is f.g.. As a submodule of R^{n-1} it is f.p. by the induction hypothesis. Hence N is f.p. by lemma 1.

Lemma 3 Let R be a left coherent ring. Then I is coflat iff $\mathrm{Ext}^n(-,I)_{\mathrm{f.p.}}=0$, $n=1$.

Proof Let $E\in$ f.p.. We may construct short exact sequences

(S_1) $O\to K_0\to P_0\xrightarrow{\varepsilon} E\to O$

(S_2) $O\to K_1\to P_1\xrightarrow{d_1} K_0\to O$

…… ……

(S_n) $O\to K_n\to P_n\xrightarrow{d_n} K_{n-1}\to O$

subsequently with $P_i\cong R^{n_i}$ being f.g. free modules, $K_0=\ker\varepsilon$ and $K_i=\ker d_i$ $(i\geqslant 1)$. Since E is f.p., K_0 is f.g. and as a submodule of $P_0=R^{n_0}$ it is f.p. following lemma 2. Similarily $K_i(i\geqslant 1)$ are all f.p.. Applying $\mathrm{Hom}_R(-,I)$ with coflat I on (S_n) yields exact sequences (T_n): $O\to\mathrm{Hom}_R(K_{n-1},I)\to\mathrm{Hom}_R(P_n,I)\to\mathrm{Hom}_R(K_n,I)$, $n\geqslant 0$.

$(K_{-1}=E)$. By coflatness of I any $f\in \mathrm{Hom}(K_n, I)$ is extendable to P_n so (T_n) may be extended to short exact sequences $O\to \mathrm{Hom}_R(K_{n-1}, I)\to \mathrm{Hom}_R(P_n, I)\to \mathrm{Hom}_R(K_n, I)\to O$. Gluing together (T_n) we obtain the exact sequence

$$O\to \mathrm{Hom}_R(E,I)\xrightarrow{\eta}\mathrm{Hom}_R(P_0,I)\xrightarrow{\delta^0}\mathrm{Hom}_R(P_1,I)\xrightarrow{\delta^1}\cdots\to \mathrm{Hom}_R(P_n,I)\xrightarrow{\delta^n}\cdots.$$

Therefore $\mathrm{Ext}^n(E,I)=\ker\delta^n/\mathrm{im}\,\delta^{n-1}=0,\ n\geqslant 1$.

Proposition 2 Let R be a coherent ring, B be any R-module and X^n be coyokes of B defined by an arbitrary coflat resolution of B. Then for every $E\in$ f. p, $\mathrm{Ext}^{n+1}(E,B)\cong \mathrm{Ext}^1(E,X^{n-1})$.

Proof The coflat resolution $O\to B\xrightarrow{\varepsilon}I^0\xrightarrow{d^0}I^1\xrightarrow{d^1}\cdots\to I^{n-1}\xrightarrow{d^{n-1}}X^{n-1}\to O$ with X^{n-1} being the $(n-1)$st coyoke may be broken into short exact sequences:

$$(S_0)\qquad O\to B\xrightarrow{\varepsilon}I^0\to X^0\to O$$

$$(S_1)\qquad O\to X^0\xrightarrow{d^0}I^1\to X^1\to O$$

$$\cdots\cdots\qquad\cdots\cdots$$

$$(S_{n-1})\qquad O\to X^{n-2}\xrightarrow{d^{n-2}}I^{n-1}\to X^{n-1}\to O.$$

Applying $\mathrm{Hom}_R(E,-)$ on (S_0) yields

$$O=\mathrm{Ext}^n(E,I^0)\to \mathrm{Ext}^n(E,X^0)\to \mathrm{Ext}^{n+1}(E,B)\to \mathrm{Ext}^{n+1}(E,I^0)=O$$

where the O on both ends are due to lemma 3. So we have $\mathrm{Ext}^{n+1}(E,I^0)\cong \mathrm{Ext}^n(E,X^0)$. Repeating the process on (S_i), $i=1,2,\cdots,n-1$, we get $\mathrm{Ext}^{n+1}(E,B)\cong \mathrm{Ext}^n(E,X^0)\cong \mathrm{Ext}^{n-1}(E,X^1)\cong\cdots\cong \mathrm{Ext}^1(E,X^{n-1})$.

Proposition 3 Let R be a coherent ring. The following conditions are equivalent for a left R-module B:

(a) $\mathrm{cfd}(B)\leqslant n$,

(b) $\mathrm{Ext}^k(-,B)_{\mathrm{f.p.}}=0$ for all $k\geqslant n+1$,

(c) $\mathrm{Ext}^{n+1}(-,B)_{\mathrm{f.p.}}=0$,

(d) every coflat resolution of B has a coflat $(n-1)$st coyoke.

Proof (a)$\Rightarrow$(b). Let $O\to B\xrightarrow{\varepsilon}I^0\xrightarrow{d^0}I^1\xrightarrow{d^1}\cdots\xrightarrow{d^{n-1}}I^n\to O$ be a coflat resolution of B. By prop. 2 for any $k>n$ and any $E\in$ f. p. $\mathrm{Ext}^k(E,B)=\mathrm{Ext}^{k-n}(E,I)$. But $\mathrm{Ext}^{k-n}(E,B)=0$ since I is coflat.

(b)$\Rightarrow$(c). Trivial.

(c)$\Rightarrow$(d). Let $O\to B\xrightarrow{\varepsilon}I^0\xrightarrow{d^0}I^1\to\cdots$ be a coflat resolution of B. Then for any $E\in$ f. p. $\mathrm{Ext}^1(E,X^{n-1})\cong\cdots\cong \mathrm{Ext}^{n+1}(E,B)=O$ by (c), which shows that $I^n=X^{n-1}$ is coflat.

(d)$\Rightarrow$(a). Trivial.

Proposition 4 If R is a left and right coherent ring, then $\mathrm{lwc}D(R)=\mathrm{rwc}D(R)$.

Proof Since the weak global dimension of R is left-right symmetrical, it is sufficient to show that $\mathrm{lwc}D(R)=\mathrm{lw}D(R)$. We show that for every $X\in {}_R\underline{M}$ $\mathrm{fd}(X)\leqslant n$ iff for every $X\in {}_R\underline{M}$ $\mathrm{cfd}(X)\leqslant n$.

Necessity. We shall show that $\mathrm{Ext}^{n+1}(-,X)_{\mathrm{f.p.}}=0$ for every $X\in {}_R\underline{M}$ and the result follows from prop. 3(c). Let $E\in$ f.p. Then there exists a free resolution of E

$$O\to K_{n-1}\xrightarrow{d^{n-1}}P_{n-1}\to\cdots\to P_1\xrightarrow{d^1}P_0\xrightarrow{\varepsilon}E\to O$$

with $P_i=R^{n_i}$ and $K_{n-1}=\ker d_{n-1}$. As in the proof of lemma 3 K_{n-1} is f. p.. By hypothesis K_{n-1} is flat, hence projective. We have $\mathrm{pd}(E)\leqslant n$ for every $E\in$ f. p., or $\mathrm{Ext}^{n+1}(E,-)=0$ as known from facts on projective dimension of modules, or $\mathrm{Ext}^{n+1}(-,X)_{\mathrm{f.p.}}=0$ for every $X\in {}_R\underline{M}$.

Sufficiency. Assume $\mathrm{cfd}(X)\leqslant n$ for every $X\in {}_R\underline{M}$. Since $X=\varinjlim_{i\in I} X_i$ with X_i being f. p. submodules of X and it is known that the functor $\varinjlim_{i\in I}$ preserves $-\otimes-$ and hence $\mathrm{Tor}_n(-,-)$, it suffices to show that for every $E\in$ f. p. $\mathrm{fd}(E)\leqslant n$. By porp. 3 $\mathrm{Ext}^{n+1}(-,X)_{\mathrm{f.p.}}=0$, or $\mathrm{Ext}^{n+1}(E,-)\leqslant O$ for every $E\in$ f. p.. Thus $\mathrm{pd}(E)\leqslant n$ for every $E\in$ f. p.. Clearly $\mathrm{fd}(E)\leqslant\mathrm{pd}(E)\leqslant n$ for every $E\in$ f. p..

Denote the left respectively right global dimension of a ring R by $\mathrm{lD}(R)$ respectively $\mathrm{lD}(R)$. $\mathrm{lD}(R)$ is known as the common value of the left projective global dimension $\mathrm{lpD}(R)$ and the left injective global dimension (or golbal codimension $\mathrm{liD}(R)$. Similarily, $\mathrm{rD}(R)=\mathrm{rpD}(R)=\mathrm{riD}(R)$.

Corollary 1 If R is a left and right Noetherian ring, then $\mathrm{lD}(R)=\mathrm{rD}(R)$.

Proof The coflat modules are injective over a Noetherian ring. Hence $\mathrm{liD}(R)=\mathrm{lwcD}(R)=\mathrm{lwD}(R)=\mathrm{rwD}(R)=\mathrm{rwcD}(R)=\mathrm{riD}(R)$.

Corollary 2 If R is a left and right coherent, left and right perfect ring, then $\mathrm{lD}(R)=\mathrm{lpD}(R)=\mathrm{rpD}(R)=\mathrm{rD}(R)$.

Proof In this case the flats coincide with the projectives.

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