

A Note on Covering Property which Implies Isocompactness*

Wu Lisheng

(Suzhou University)

In this note we defined a covering property which implies isocompactness and generalizes both the weak $[\aleph_1, \infty)^r$ -refinability and the superpurity.

In recent years a number of weak covering properties which imply isocompactness were discovered. For example, the $\theta L^{[1]}$, the purity and superpurity^[2] and the weak $[\aleph_1, \infty)^r$ -refinability^[3]. These properties all generalize the weak $\delta\theta$ -refinability. Now we shall define a new covering property—pseudo $[\aleph_1, \infty)^r$ -refinability, it is weaker than both superpurity and weak $[\aleph_1, \infty)^r$ -refinability, but still implies isocompactness.

If X is a topological space, \mathcal{U} is a collection of subsets of X , we denote $\cup\{U: U \in \mathcal{U}\}$ by \mathcal{U}^* and denote $\{\mathcal{V}^*: \mathcal{V} \subseteq \mathcal{U}, |\mathcal{V}| < \delta\}$ by $\mathcal{U}^{<\delta}$, where δ is a cardinal number.

Definition 1 If X is a space, γ is a cardinal number, and \mathcal{U} is an open cover of X , a pseudo- $[\gamma]$ -refinement of \mathcal{U} is a collection of collections of subsets $\{\mathcal{B}_\lambda: \lambda < \tau\}$ such that: (1) $|\{\mathcal{B}_\lambda: \lambda < \tau\}| = \tau < \gamma$, (2) $\cup\{\mathcal{B}_\lambda^*: \lambda < \tau\} = X$, (3) for each $\lambda < \tau$ and each $D \in \mathcal{B}_\lambda$, D is an open subset of \mathcal{B}_λ^* , and (4) the collection $\{\text{st}(x, \mathcal{B}_\lambda): \lambda < \tau, x \in \mathcal{B}_\lambda^*\}$ is a refinement of $\mathcal{U}^{<\gamma}$.

Definition 2 Let α and β be cardinal numbers. A space X is called pseudo $[a, \beta]^r$ -refinable, if and only if for every regular cardinal γ such that $a \leq \gamma \leq \beta$, if \mathcal{U} is an open cover of X such that $|\mathcal{U}| = \gamma$, then there is a pseudo $[\gamma]$ -refinement of \mathcal{U} . A space is pseudo $[a, \infty)^r$ -refinable if and only if it is pseudo $[a, \beta]^r$ -refinable for all $\beta \geq a$.

Theorem 1 If X is pseudo $[a, \beta]^r$ -refinable, $A \subseteq X$ is a closed subset, then A is pseudo $[a, \beta]^r$ -refinable.

Theorem 2 If $X = \cup\{X_n: n < \omega\}$, each X_n is pseudo $[a, \beta]^r$ -refinable, $a \geq \aleph_1$, then X is pseudo $[a, \beta]^r$ -refinable.

Corollary 1 If X is pseudo $[a, \beta]^r$ -refinable, $A \subseteq X$ is a F_σ -subset, then A is pseudo $[a, \beta]^r$ -refinable.

Theorem 3 If X is weak $[\aleph_1, \infty)^r$ -refinable, then X is pseudo $[\aleph_1, \infty)^r$ -refinable.

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Lemma 1 X is superpure space if and only if for every open cover \mathcal{U} , there exists a collection of collections of subsets of X $\{\mathcal{D}_n: n < \omega\}$ such that: (1) $\bigcup\{\mathcal{D}_n^*: n < \omega\} = X$, (2) for each $n < \omega$ and each $D \in \mathcal{D}_n$, D is an open subset of \mathcal{D}_n^* , and (3) $\{\text{st}(x, \mathcal{D}_n): n < \omega, x \in \mathcal{D}_n^*\}$ is a refinement of $\mathcal{U}^{<\omega}$.

Theorem 4 If X is superpure, then X is pseudo $[\aleph_1, \infty)$ -refinable.

Lemma 2 Suppose that X is a space and μ is an infinite cardinal such that every subset of X of cardinality $> \mu$ has a 2-limit point and every wellordered increasing open cover \mathcal{U} of X of regular cardinality $> \mu$ has a pseudo $[|\mathcal{U}|]$ -refinement, then no free closed ultrafilter on X has the μ -intersection property.

Proof It is a version of the proof of [3, Theorem 3.1]. We state here only for the sake of completeness.

Suppose \mathcal{F} is a free closed ultrafilter on X having the μ -intersection property. Let γ be the smallest cardinal such that there exists $\mathcal{F}' \subseteq \mathcal{F}$ with $|\mathcal{F}'| = \gamma$ and $\bigcap \mathcal{F}' = \emptyset$. Then $\mu < \gamma$. Let $\mathcal{F}' = \{F_\alpha: \alpha < \gamma\}$ where $\bigcap \mathcal{F}' = \emptyset$. Let $B_\alpha = \bigcap \{F_\delta: \delta < \alpha\}$ for all $\alpha < \gamma$. Then $\mathcal{U} = \{X - B_\alpha: \alpha < \gamma\}$ is a wellordered increasing open cover of X , and γ is a regular cardinal. (See [3, Theorem 3.1 proof].)

By the hypothesis, \mathcal{U} has a pseudo $[|\mathcal{U}|]$ -refinement $\{\mathcal{D}_\lambda: \lambda < \tau\}$. From $\bigcup\{\mathcal{D}_\lambda^*: \lambda < \tau\} = X$ and $\tau < \gamma$, it follows that there exists some $\lambda_0 < \tau$ such that $\mathcal{D}_{\lambda_0}^*$ intersects each element of \mathcal{F} . For each $D \in \mathcal{D}_{\lambda_0}$, $D = \mathcal{D}_{\lambda_0}^* \cap V_D$, where V_D is an open subset of X . Let $\mathcal{V} = \{V_D: D \in \mathcal{D}_{\lambda_0}\}$, then for some $M \in \mathcal{F}$, $M \subseteq \mathcal{V}^*$, and there exists a discrete subset A of X such that $M \cap \mathcal{D}_{\lambda_0}^* \subseteq \text{st}(A, \mathcal{V})$, thus $M \cap \mathcal{D}_{\lambda_0}^* \subseteq \text{st}(A, \mathcal{V}) \cap \mathcal{D}_{\lambda_0}^* = \text{st}(A, \mathcal{D}_{\lambda_0})$. A has no 2-limit points, so $|A| \leq \mu$. For each $a \in A$, take a $\mathcal{U}_a \subseteq \mathcal{U}$ such that $|\mathcal{U}_a| < \gamma$ and $\text{st}(a, \mathcal{D}_{\lambda_0}) \subseteq \mathcal{U}_a^*$. Let $\mathcal{B} = \bigcup\{\mathcal{U}_a: a \in A\}$, then $\mathcal{B} \subseteq \mathcal{U}$, $|\mathcal{B}| < \gamma$, $M \cap \mathcal{D}_{\lambda_0}^* \subseteq \mathcal{B}^*$. But $X - \mathcal{B}^* = \bigcap\{X - B: B \in \mathcal{B}\} \in \mathcal{F}$, $(X - \mathcal{B}^*) \cap M \in \mathcal{F}$, $[(X - \mathcal{B}^*) \cap M] \cap \mathcal{D}_{\lambda_0}^* = \emptyset$, this is a contradiction.

Theorem 5 Each pseudo $[\aleph_1, \infty)$ -refinable space is isocompact.

Theorem 6 Suppose X is \aleph_1 -compact and T_1 and pseudo $[\aleph_1, \infty)$ -refinable. Then X is closed-complete.

Theorem 7 Suppose X is countably compact and T_2 . Then X is metrizable if and only if X is pseudo $[\aleph_1, \infty)$ -refinable and has a θ -diagonal.

Theorem 8 If X is a hereditary \aleph_1 -compact T_1 -space which is superpure, then X is Lindelöf.

References

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