第4卷 第4期
1984年10月

数 学 研 究 与 评 论
JOURNAL OF MATHEMATICAL
RESEARCH AND EXPOSITION

Vol.4 No.4
Oct. 1984

The Simple Proof of Two Lemmas in "Transversal Mappings and Flows"*

Lin Zhen Sheng (林振声)

(Fuzhou University)

In this note we give the new proof of perturbation lemma and tangent perturbation lemma of R. Abraham and J. Robbin in "Transversal Mappings and Flows" (§32, Chapter 7.). The following proof is simpler than the original one.

§1. Perturbation Lemma Suppose that X is the compact C^∞ manifold, $F^0\in\Gamma^r(J_X)$, S^0 is the flow of F^0, $\Gamma^r(J_X)(r\geqslant 1)$ is the set of all C^r vectorfields on X, γ is the closed orbit of F^0 with period $J>0$. If $\dot{x}\in T_xX$, where T_xX is the tangent space of X at the point x, then we have $\eta\in\Gamma^r(J_X)$ such that

$$\frac{d}{d\lambda}(S_J^\lambda(x))_{\lambda=0}=\dot{x},$$

where S^λ is the flow of $F^\lambda=F^0+\lambda\eta$, $\lambda\in R$, R is the real axis.

Proof By the meaning of flow we have

$$S_t^\lambda(x)=x+\int_0^t(F^0(S_s^\lambda(x))+\lambda\eta(S_s^\lambda(x)))ds$$

$$\frac{d}{d\lambda}S_t^\lambda(x))=\int_0^t\left(\frac{\partial F^0(S_s^\lambda(x))}{\partial y}+\lambda\frac{\partial\eta(S_s^\lambda(x))}{\partial y}\right)\frac{d}{d\lambda}(S_s^\lambda(x))ds+\int_0^t\eta(S_s^\lambda(x))ds \tag{1}$$

i.e., $\frac{d}{d\lambda}(S_t^\lambda(x))_{\lambda=0}$ satisfies the linear system

$$\frac{dz}{dt}=\left(\frac{\partial F^0(S_t^0(x))}{\partial y}\right)z+\eta(S_t^0(x)),\quad y=S_t^0(x).$$

On the other hand $T_xS_t^0(x)=\left(\frac{\partial S_t^0(x)}{\partial x}\right)=Y(t), Y(0)=E$, is the fundamental matrix of the linear system

$$\frac{dy}{dt}=\left(\frac{\partial F^0(S_t^0(x))}{\partial y}\right)y, \tag{2}$$

so we can express

$$\frac{d}{d\lambda}(S_t^\lambda(x))_{\lambda=0}=\int_0^t T_xS_t^0(x)(T_xS_s^0(x))^{-1}\eta(S_s^0(x))ds. \tag{3}$$

Let us take $\eta(y)=g(s)T_xS_s^0(x)(T_xS_J^0(x))^{-1}\dot{x}$, when $y=S_s^0(x)$, where g is the mapping such that

$$g:\ R\to R,\ D^ig(J)=D^ig(0),\ i=0,1,2,\cdots,r,\quad \int_0^J g(s)ds=1.$$

The η is defined on γ. It can be found in [1] p. 110 how to extend the η from γ to all X. Therefore we have $\frac{d}{d\lambda}(S_J^\lambda(x))_{\lambda=0}=\dot{x}$.

*Received Feb. 17, 1983.

The Lemma is proved completely.

Corollary If $\eta|\gamma\equiv 0$, then $\frac{d}{d\lambda}(S_t^\lambda(x))_{\lambda=0}=0$.

proof By formula (3).

§2. Tangent perturbation Lemma Let $F^0\in\Gamma^r(J_X)$, $(r\geqslant 1)$, γ is the closed orbit of F^0 with period $J>0$, U is the nerghborhood of γ in X, $x\in\gamma$, and $A\in L(T_xX,T_xX)$ which is the set of all linear maps from T_xX into itself with the natural structure of finite dimensional real vector space. Then if $A\ F^0(x)=0$, there exists $\eta\in\ \Gamma^r(J_X)$ such that

(I). $\eta|\gamma\equiv 0$; (II). $\eta|X\backslash U\equiv 0$;

(III). If for $\lambda\in R$, S^λ is the flow of $F^\lambda=F^0+\lambda\eta$, then

$$\frac{d}{d\lambda}(T_xS_J^\lambda(x))_{\lambda=0}=A.$$

Proof Suppose that there is $\eta\in\Gamma^r(J_X)$ to satisfy (I) and (II), Then we have

$$S_t^\lambda(x)=x+\int_0^t(F^0(S_s^\lambda(x))+\lambda\eta(S_s^\lambda(x)))ds,$$

$$T_xS_t^\lambda(x)=E+\int_0^t\left(\frac{\partial F^0(S_s^\lambda(x))}{\partial y}+\lambda\frac{\partial\eta(S_s^0(x))}{\partial y}\right)T_xS_s^0(x)ds,$$

$$\frac{d}{d\lambda}(T_xS_t^\lambda(x))_{\lambda=0}=\int_0^t\frac{\partial F^0(S_s^0(x))}{\partial y}\frac{d}{d\lambda}(T_xS_s^\lambda(x))_{\lambda=0}+\frac{\partial\eta(S_s^0(x))}{\partial y}T_xS_s^0(x))ds$$
$$+\int_0^t\left(\sum_{k=1}^n\frac{\partial^2F^0(S_s^0(x))}{\partial y_k\partial y_i}\frac{d}{d\lambda}(S_s^\lambda(x))_{\lambda=0}T_xS_s^0(x)ds.\right.$$

By the Corollary of above Lemma $\frac{d}{d\lambda}(S_s^\lambda(x)_{\lambda=0}=0$, we have

$$\frac{d}{d\lambda}(T_xS_t^\lambda(x))_{\lambda=0}=\int_0^t\frac{\partial F^0(S_s^0(x))}{\partial y}\frac{d}{d\lambda}(T_xS_s^\lambda(x))_{\lambda=0}ds+\int_0^t\frac{\partial\eta(S_s^0(x))}{\partial y}T_xS_s^0(x)ds,$$

i. e., $\frac{d}{d\lambda}(T_xS_t^\lambda(x))_{\lambda=0}$ satisfies the linear system

$$\frac{dz}{dt}=\left(\frac{\partial F^0(S_t^0(x))}{\partial y}\right)z+\frac{\partial\eta(S_t^0(x))}{\partial y}T_xS_t^0(x),$$

so we can express

$$\frac{d}{d\lambda}(T_xS^\lambda(x))_{\lambda=0}=\int_0^JT_xS_t^0(x)(T_xS_s^0(x))^{-1}\frac{\partial\eta(S_s^0(x))}{\partial y}T_xS_s^0(x)ds.$$

Let us define

$$T_x\eta(y)=\frac{\partial}{\partial y}\eta(S_s^0(x))T_xS_s^0(x)=g(s)T_xS_s^0(x)(T_xS_J^0(x))^{-1}A,$$

when $y=S_s^0(x)$, the meaning of $g(s)$ is the same as that in above lemma. Hence

$$\frac{d}{d\lambda}(T_xS_J^\lambda(x))_{\lambda=0}=\int_0^tg(s)Ads=A.$$

The construction of η is sketched in [1], p. 112: It does not repeat here.

Reference

[1] Abraham, R. and Robbin, J., Transversal Mappings and Flows, W. A. Benjamin Inc. New York (1967).