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# The Exponential Stability in the Large of Discrete Systems*

*Xu Daoyi* (徐道义)

(*Mianyang Teachers' College, Sichuan*)

We consider a discrete system

$$X(t_{k+1}) = A[t_k, x(t_k)]X(t_k), \tag{1}$$

where $x(t_k) \in R^n$ is the state of system (1), $t_k \in J = (T, +\infty)$ is discrete time, where $k$ is an integer and $T$ is a number or a symbol $-\infty$; and $t_k \to +\infty$ when $k \to +\infty$. The matrix function $A[t_k, x(t_k)]$: $J \times R^n \to R^{n^2}$ is bounded, the elements of $A[t_k, x(t_k)]$ are denoted by $a_{ij}[t_k, x(t_k)]$ and $a_{ij} \triangleq \sup_{x \in R^n, t_k \in J} a_{ij}[t_k, x(t_k)]$ $(i, j = 1, \cdots, n)$.

**Definition** The equilibrium $x^* = 0$ of the system (1) is exponentially stable in the large if there exist numbers $\Pi \geqslant 1$, $\pi > 0$, which do not depend on the initial condition $(t_0, x_0)$, such that for all $(t_0, x_0) \in J \times R^n$.

$$\|x(t_k;\ t_0, x_0)\| \leqslant \Pi \|x_0\| e^{-\pi(t_k - t_0)} \quad \forall t_k \in J_0 = [t_0 + \infty) \tag{2}$$

where $x(t_k; t_0, x_0)$ is a solution of system (1) satisfying $x(t_0; t_0, x_0) = x_0$, *and* $\|\cdot\|$ *refers* to the Euclidean norm.

For the main result of this paper, we make use of the following simple

**Lemma** If the spectral radius of an $m \times m$ constant matrix $B$ is less than one, then there exist numbers $\Pi_1 > 0$, $\pi_1 > 0$ such that

$$\|B^k\| \leqslant \Pi_1^{-\pi_1(t_k - t_0)} \qquad \forall t_k \in J_0,\ t_0 \in J.$$

**Proof** Since the matrix $B$ is similar to an $m \times m$ matrix $J$ in Jordan form by the knowledge in linear algebra, *i.e.* there is an invertible $m \times m$ matrix $S$ such that $B = S^{-1}JS$ where $J = \mathrm{diag}[J_1, \cdots, J_r]$ and

$$J_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \lambda_i & 1 \\ 0 & & & \lambda_i \end{pmatrix} \qquad i = 1, \cdots, r, \quad \sum_{i=1}^{r} p_i = m,$$

$\lambda_i$ being a characteristic root of $B$.

Again since

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$$J_i^k=\begin{pmatrix} \lambda_i^k & p_i\lambda_i^{k-1}\cdots c_k^{p_i}\lambda^{k-p_i} \\ & \lambda_i^k \ddots \vdots \\ & \ddots \; p_i\lambda_i^{k-1} \\ 0 & \lambda_i^k \end{pmatrix}$$

where $C_k^{p_i}=\dfrac{k(k-1)\cdots(k-p_i+1)}{p_i!}$ $(k\geqslant p_i)$; $C_k^{p_i}=0$ $(k<p_i)$, and

$$B^k=S^{-1}J^kS=S^{-1}\mathrm{diag}(J_1^k,\cdots,J_r^k)S.$$

Therefore,

$$\begin{aligned}\|B^k\|&\leqslant\|S^{-1}\|\|S\|\|\mathrm{diag}(J_1^k,\cdots,J_r^k)\|\\ &\leqslant\|S^{-1}\|\|S\|mc_k\lambda_M^{k-m}\quad(\lambda_M=\max_{1\leqslant i\leqslant r}|\lambda_i|<1;\ c_k=\max_{1\leqslant i\leqslant m}c_k^i)\\ &=\frac{mc_k\|S^{-1}\|\|S\|}{\lambda_M^{m-t_0/T_M}}\lambda_M^{\frac{1}{T_M}(T_Mk-t_0)}\qquad(T_M=\sup_k(t_{k+1}-t_k))\\ &\leqslant\frac{mc_k\|S^{-1}\|\|S\|}{\lambda_M^{m-t_0/T_M}}\lambda_M^{\frac{1}{T_M}(t_k-t_0)}\leqslant\Pi_1e^{-\pi_1(t_k-t_0)},\end{aligned}$$

where $\Pi_1\geqslant\dfrac{mc_k\|S^{-1}\|\|S\|}{\lambda_M^{m-(t_k+t_0)/2T_M}}>0$, $\pi_1=\dfrac{1}{2T_M}\ln\dfrac{1}{\lambda_M}>0$. The proof of Lemma is complete.

We now state and prove our main result.

**Theorem 1** The equilibrium $x^*=0$ of the system (1) is exponentially stable in the large if there exists a non-negative vector function $V$: $J\times R^n\to R^m$, number $\eta_1>0$, $\eta_2>0$, and an $m\times m$ constant matrix $B$, of which the spectral radius is less than one, such that

$$\eta_1\|x\|^a\leqslant\|V\|\leqslant\eta_2\|x\|^a\qquad\forall(t_k,x)\in J_0\times R^n,\tag{3}$$

$$V[t_{k+1},x(t_{k+1};t_0,x_0)]\leqslant BV[t_k,x(t_k;t_0,x_0)],\tag{4}$$

for all $(t_0,x_0)\in J\times R^n$.

**Proof** From (4), it follows that

$$V[t_k,x(t_k;t_0,x_0)]\leqslant B^kV(t_0,x_0).$$

By means of Lemma and the property that $V$ is non-negative, we have

$$\|V(t_k,x(t_k;t_0,x_0)]\|\leqslant\|B^k\|\|V(t_0,x_0)\|\leqslant\Pi_1e^{-\pi_1(t_k-t_0)}\|V(t_0,x_0)\|.$$

From (3), we get (2) with $\Pi=\max[1,(\eta_1^{-1}\eta_2\Pi_1)^{1/a}]$, $\pi=\dfrac{\pi_1}{a}$. This completes the proof.

**Remark** In theorem 1, we get the sufficient condition in Theorem 4.11 of [1] when $a=2$ and $B=\eta_3$ (*i.e.* $m=1$), which is a positive number, is less than one (In Theorem 4.11 of [1], only the sufficient condition is true for system (1) in this paper). In [1] this condition is proved as follows:

From $V[t_{k+1}, x(t_{k+1}, t_0, x_0)] \leqslant n_3 V[t_k, x(t_k; t_0, x_0)]$, it follows that

$$\begin{aligned} V(t_{k+1}, x(t_{k+1}; t_0, x_0)] &\leqslant \Pi_1 V[t_k, x(t_k; t_0, x_0)] \exp[-2\pi T_M)] \\ &\leqslant \Pi_1 V[t_k, x(t_k, t_0 x_0] \exp[-2\pi(t_{k+1}-t_k)], \end{aligned} \tag{5}$$

where $\eta_3 = \Pi_1 \exp(-2\pi T_M)$, $T_M = \sup_k (t_{k+1} - t_k)$. From (3) and (5), we get (2) with $\Pi = (\Pi_1 \eta_1^{-1} \eta_2)^{1/2}$.

In fact, here $\Pi$ should be $(\Pi_1^k \eta_1^{-1} \eta_2)^{1/2}$, and it is not a constant numbre. Therefore, this proof in [1] is wrong.

With Theorem 1 at our disposal, we can prove the desired result as follows:

**Theorem 2** The equilibrium $x^* = 0$ of the system (1) is exponentially stable in the large if there exist $d_i > 0$ $(i = 1, \cdots, n)$ such that the spectral radius of the non-negative matrix $B = (b_{pq})_{m \times m}$ is less than one, where

$$b_{pq} = \max_{j = N_{q-1}+1, \cdots, N_q} \sum_{i = N_{p-1}+1}^{N_p} d_i d_j^{-1} a_{ij}, \quad (p, q = 1, \cdots, m; 0 = N_0 < N_1 < \cdots < N_m = n).$$

Proof Consider a function $V$: $R^n \to R^m$ defined by

$$V = [V_1, \cdots, V_m]^T; \quad V_p = \sum_{i = N_{p-1}+1}^{N_p} d_i |x_i|, \quad p = 1, \cdots, m.$$

It is evident that the function $V$ satisfies the condition(3). To show condition (4), we com pute $V_p[x(t_{k+1}; t_0, x_0)]$ $(p = 1, \cdots, m)$ using (1) as follows.

$$\begin{aligned} V_p[x(t_{k+1}; t_0, x_0] &= \sum_{i = N_{p-1}+1}^{N_p} d_i |x_i(t_{k+1}; t_0, x_0)| \\ &= \sum_{i = N_{p-1}+1}^{N_p} d_i \left| \sum_{j=1}^{n} a_{ij}[t_k, x(t_k; t_0, x_0)] x_j(t_k; t_0, x_0) \right| \\ &\leqslant \sum_{i = N_{p-1}+1}^{N_p} d_i \sum_{j=1}^{n} a_{ij} |x_j(t_k; t_0, x_0| = \sum_{j=1}^{n} \sum_{i = N_{p-1}+1}^{N_p} d_i a_{ij} |x_j(t_k; t_0, x_0)| \\ &= \left( \sum_{j=1}^{N_1} + \cdots + \sum_{j = N_{q-1}+1}^{N_q} + \cdots + \sum_{j = N_{m-1}+1}^{n} \right) \sum_{i = N_{p-1}+1}^{N_p} d_i a_{ij} |x_j(t_k; t_0, x_0)| \\ &= \left( \sum_{j=1}^{N_1} \sum_{i = N_{p-1}+1}^{N_p} + \cdots + \sum_{j = N_{q-1}+1}^{N_q} \sum_{i = N_{p-1}+1}^{N_p} + \cdots + \sum_{j = N_{m-1}+1}^{n} \sum_{i = N_{p-1}+1}^{N_p} \right) \\ &\quad \cdot [d_i a_{ij} d_j^{-1} d_j |x_j(t_k; t_0, x_0)|]. \end{aligned}$$

Let $b_{pq} = \max_{j = N_{q-1}+1, \cdots, N_q} \sum_{i = N_{p-1}+1}^{N_p} d_i d_j^{-1} a_{ij}$, $(p, q = 1, \cdots, m)$, then

$$V_p[x(t_{k+1}; t_0, x_0)] = \sum_{q=1}^{m} b_{pq} V_q[x(t_k; t_0 x_0)], \quad (p = 1, \cdots, m)$$

i.e.

$$V[x(t_{k+1}; t_0, x_0)] \leqslant BV[x(t_k; t_0, x_0)]. \tag{6}$$

Since the spectral radius of $B$ is less than one, then we get (2) by applying Theorem 1 to (6), and the proof of Theorem 2 is complete.

When $d_i=1$ $(i=1,\cdots,n)$ and $m=n$ in Theorem 2, we get the following interesting.

**Corollary** The equilibrium $x^*=0$ of the system (1) is exponentially stable in the large if the spectral radius of matrix $A^*=(a_{ij})_{n\times n}$ is less than one.

In Theorem 2, when the first kind norm of matrix $B$ (See[2]) is less than one and $m=n$, we get Theorem 4.12 in [1].

## References

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# 离散系统的全局指数稳定性

徐道义

(四川绵阳师专)

## 摘要

本文给出了离散系统全局指数稳定性的两个充分条件，这些条件较文[1]的相应结果更为广泛且便于应用。文中还订正了文[1]在证明其结果时的一点错误。