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On a Proof of Roth's Theorem*

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We denote by $M_{n,m}(F)$ the set of all $n\times m$ matrices over the field F and by $M_n(F)$ the set of all $n\times n$ matrices over the field F. W. E. Roth has shown the following theorem in 1952, [1].

Theorem Let $A\in M_n(F)$, $B\in M_m(F)$ and $C\in M_{n,m}(F)$, then the matrix equation

$$AX-YB=C \tag{1}$$

has a solution X, $Y\in M_{n,m}(F)$ if and only if the matrices

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \tag{2}$$

are eguivalent (equal rank).

Roth's proof were based on Jordan canonical forms. H. Flanders and H. K. Wimmer have given new proof of Roth's theorem by means of linear transformations and dimension argument. In this note, we shall give a proof of the result by matrix technique.

Proof of Roth's Theorem. Let the equation (1) has a solution X, $Y\in M_{n,m}(F)$, then

$$\begin{pmatrix} I_{(n)} & -Y \\ 0 & I_{(m)} \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I_{(n)} & X \\ 0 & I_{(m)} \end{pmatrix}=\begin{pmatrix} A & AX-YB \\ 0 & B \end{pmatrix}=\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.$$

Hence (1) imply equal rank in (2).

For the converse, let two matrices in (2) are equivalent. We denote by rankA the rank of matrix A, and assume that $\operatorname{rank}A=r$ and $\operatorname{rank}B=s$, then there are non-singular matrices P, $Q\in M_n(F)$ and R, $S\in M_m(F)$ such that

$$PAQ=\begin{pmatrix} I_{(r)} & 0 \\ 0 & 0 \end{pmatrix}. \quad \text{and} \quad RBS=\begin{pmatrix} I_{(s)} & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus

$$\begin{pmatrix} P & 0 \\ 0 & R \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} Q & 0 \\ 0 & S \end{pmatrix}=\begin{pmatrix} I_{(r)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{(s)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} P & 0 \\ 0 & R \end{pmatrix}\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}\begin{pmatrix} Q & 0 \\ 0 & S \end{pmatrix}=\begin{pmatrix} PAQ & PCS \\ 0 & RBS \end{pmatrix}=\begin{pmatrix} I_{(r)} & 0 & C_{11} & C_{12} \\ 0 & 0 & C_{21} & C_{22} \\ 0 & 0 & I_{(s)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

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where

$$PCS=\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}.$$

But

$$\begin{pmatrix} I_{(r)} & 0 & -C_{11} & 0 \\ 0 & I_{(n-r)} & -C_{21} & 0 \\ 0 & 0 & I_{(s)} & 0 \\ 0 & 0 & 0 & I_{(m-s)} \end{pmatrix}\begin{pmatrix} I_{(r)} & 0 & C_{11} & C_{12} \\ 0 & 0 & C_{21} & C_{22} \\ 0 & 0 & I_{(s)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} I_{(r)} & 0 & 0 & -C_{12} \\ 0 & I_{(n-r)} & 0 & 0 \\ 0 & 0 & I_{(s)} & 0 \\ 0 & 0 & 0 & I_{(m-s)} \end{pmatrix}=\begin{pmatrix} I_{(r)} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{22} \\ 0 & 0 & I_{(s)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

hence $C_{22}=0$ since two matrices in (2) are equal rank. We denote respectively by J and K the matrices

$$\begin{pmatrix} C_{11} & 0 \\ C_{21} & 0 \end{pmatrix} \text{and} \begin{pmatrix} 0 & C_{12} \\ 0 & 0 \end{pmatrix},$$

then

$$\begin{pmatrix} I_{(n)} & -J \\ 0 & I_{(m)} \end{pmatrix}\begin{pmatrix} P & 0 \\ 0 & R \end{pmatrix}\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}\begin{pmatrix} Q & 0 \\ 0 & S \end{pmatrix}\begin{pmatrix} I_{(n)} & -K \\ 0 & I_{(m)} \end{pmatrix}=\begin{pmatrix} P & 0 \\ 0 & R \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} Q & 0 \\ 0 & S \end{pmatrix}.$$

Hence

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}=\begin{pmatrix} P^{-1} & 0 \\ 0 & R^{-1} \end{pmatrix}\begin{pmatrix} I_{(n)} & J \\ 0 & I_{(m)} \end{pmatrix}\begin{pmatrix} P & 0 \\ 0 & R \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} Q & 0 \\ 0 & S \end{pmatrix}\begin{pmatrix} I_{(n)} & K \\ 0 & I_{(m)} \end{pmatrix}\begin{pmatrix} Q^{-1} & 0 \\ 0 & S^{-1} \end{pmatrix}$$

$$=\begin{pmatrix} A & AQKS^{-1}+P^{-1}JRB \\ 0 & B \end{pmatrix},$$

therefore

$$C=A(QKS^{-1})-(-P^{-1}JR)B.$$

Thus

$$X=QKS^{-1} \quad \text{and} \quad Y=-P^{-1}JR$$

is a solution of the equation (1). This proves Roth's theorem.

References

[1] Roth, W. E., The equations $AX-YB==C$ and $AX-XB=C$ in matrices, *Proc. Amer. Soc.*, 3(1952), pp. 392-396.

[2] Franders. H. and Wimmer, H. K., On the matrix equation $AX-XB=C$ and $AX-YB=C$ SIAM., *J. Appl. Math.*, 32 (1977) pp. 707-710.