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Generalizations of C. T. Yang's Theorem and K. Fan's Theorem*

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C. T. Yang's theorem concerning derived sets and K. Fan's theorem concerning connectivity in general topology have been extended to fuzzy topological spaces[1-2] and to the theory of topological molecular lattices and generalized topological molecular lattices[3-4]. In this paper, they will be further extended to the more general theory of POINTWISE TOPOLOGY ON COMPLETELY DISTRIBUTIVE LATTICES[7].

§1 On the Structure of Completely Distributive Lattices

Throughout this paper, L will be complete lattice. In this section we deal with the discussion of the structure of completely distributive lattices.

Definition 1 Suppose that $a\in L$, $A\subset L$. Then A is called a maximal family of a if $A\neq\varnothing$ and

(i) $\inf A=a$.

(ii) $\forall B\subset L$, $\inf B\leqslant a$ implies that $\forall x\in A$ there exists $y\in B$ such that $y\leqslant x$.

It is easy to verify that the union of maximal families of a is a maximal family of a as well. Hence if a has a maximal family, then a has a greatest one, it will be denoted by $\alpha(a)$.

Examples (1) Let $L=[0,1]$, $a\in[0,1)$, then $\alpha(a)=(a,1]$ and $\alpha(1)=\{1\}$; Moreover, suppose that D is dense in $[0,1]$ and $a\in[0,1)$, then $D\cap\alpha(a)$ is still a maximal family of a.

(2) Let $L=\mathscr{P}(X)$, $E\subset X$, $E\neq X$, then

$$\alpha(E)=\{X\backslash\{e\}\mid e\notin E\}\cup\{X\}$$

and $\alpha(X)=\{X\}$.

Definition 2 Suppose that $a\in L$, $B\subset L$, then B is called a minimal family of a if $B\neq\varnothing$ and

(i) $\sup B=a$.

(ii) $\forall A\subset L$, $\sup A\geqslant a$ implies that $\forall x\in B$ there exists $y\in A$ such that $y\geqslant x$.

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Similarly, when a has a minimal family, it has a greatest one, denoted $\beta(a)$.

L is said to be completely distributive if the following condition is satisfied:

$$\bigwedge_{i\in I}\Big(\bigvee_{j\in J_i} a_{i,j}\Big)=\bigvee_{f\in \times J_i}\Big(\bigwedge_{i\in I} a_{i,f(i)}\Big).$$

In [5] we have proved the following

Theorem 1 Suppose that L is a complete lattice, then the following conditions are equivalent to each other:

(1) L is completely distributive.

(2) $\forall a\in L$, a has a maximal family.

(3) $\forall a\in L$, a has a minimal family.

An element a of L is said to be $\vee$-irreducible if $a\leqslant b\vee c$ implies that $a\leqslant b$ or $a\leqslant c$, where b, $c\in L$. By means of the theory of maximal families we have proved in [5] that every element of a completely distributive lattice L can be expressed as an union of $\vee$-irreducible elements of L. Let $\beta(a)$ be the greatest minimal family of a, one readily checks that

$$\beta^*(a)=\{x\in L\,|\,x \text{ is } \vee\text{-irreducible and } x\leqslant y \text{ for some } y\in\beta(a)\}$$

is a minimal family of a consisting of $\vee$-irreducible elements, such a family will be called a standard minimal family of a. Then we have

Theorem 2 Suppose that L is a complete lattice, then L is completely distributive iff $\forall a\in L$, a has a standard minimal family.

Suppose that L is a comletely distributive lattice, and $M\subset L$ is the subset consisting of all non-zero $\vee$-irreducible elements, then elements of M are similar to points of a fixed set X, and we prefer to endow them a vivid name—molecules. L will then be called a molecular lattice and denoted by $L(M)$.

§2 Topological Molecular Lattices

Definition 3 Let $L(M)$ be a molecular lattice, η is a subset of L. If $0,1\in\eta$ and η is closed under the operations of finite unions and arbitrary intersections, then it will be called a co-topology on L and $(L(M),\eta)$ is called a topological molecular lattice, or briefly, TML. Moreover, for every $A\subset L$, the closure A^- of A is the intersection of all the elements of η which contain A.

The nomenclature "Topological molecular lattice" was firstly used by the author to mean certain mathematical objects which are similar to the one given above but restricted by some additional conditions[3]. From now on, we would rather by this nomenclature to mean the wider class of objects described in definition 3.

Examples (1) Let $(X,\mathscr{U})$ be a topological space, then $(\mathscr{P}(X),\eta)$ is a TML, where $\eta=\mathscr{U}'=\{U'\,|\,U\in\mathscr{U}\}$ and moleculas are single point sets of $\mathscr{P}(X)$.

(2) Let (X, δ) be a fuzzy topological space, then (I^X, η) is a TML, where $\eta=\delta'$, and molecules are fuzzy points.

(3) Let (L,δ) be a topological molecular lattice in the narrow sense of [3], then (L, η) is a TML where $\eta=\delta'$

(4) Let $(L(M),\delta)$ be a generalized topological moleculaт lattice, then $(L(M),\eta)$ is a TML where $\eta=\delta'$ and M is the set consisting of all non-zero $\vee$-irreducible elements of L.

Definition 4 Let $(L(M), \eta)$ be a TML, $a\in M$, $P\in\eta$. If $a\not\leqslant P$, then P is called a remote-neighborhood of a, or briefly R-neighborhood of a. The set of all R-neighborhoods of a will be denoted by $\eta(a)$.

Clearly, $\eta(a)$ is an ideal base.

Definition 5 Let $(L(M), \eta)$ be a TML, $A\in L$, $a\in M$. If for every $P\in\eta(a)$ we have $A\not\leqslant P$, then a is called an adherence point of A; If a is an adherence point of A and $a\not\leqslant A$, or $a\leqslant A$ and for every $b\in M$ satisfyiug $a\leqslant b\leqslant A$ we have $A\not\leqslant b\vee P$, where $P\in\eta(a)$, then a is called an accumulation point of A. The union of all accumulation points of A is called the derived element of A and will be denoted by A^d.

One easily checks that these concepts coincide with the old ones in general topology in case $L=\mathscr{P}(X)$ and the corresponding ones in case $L=[0,1]^X$.

Theorem 3 Let $(L(M), \eta)$ be a TML, $A\in L$, $a\in M$, then

(1) a is an adherence point of A iff $a\leqslant A^-$.

(2) $A^-=\{a\in M \mid a \text{ is an adherence point of } A\}$

(3) $A^-=A\vee A^d$, $(A^d)^-\leqslant A^-$.

Proof On account of the definition of closures and R-neighborhoods it follows that

$$\begin{aligned} a\leqslant A^- &\Leftrightarrow \text{if } P \text{ is closed and } A\leqslant P, \text{ then } a\leqslant P \\ &\Leftrightarrow \text{if } P \text{ is closed and } a\not\leqslant P, \text{ then } A\not\leqslant P \\ &\Leftrightarrow \text{if } P\in\eta(a), \text{ then } A\not\leqslant P. \end{aligned}$$

By definition 5, statement (1) follows. Now (2) follows from (1) and definition 3 and (3) is then evident.

Definition 6 Let $(L(M), \eta)$ be a TML and D a dericted set, then the mapping $S: D\to M$ is called a molecular net and will be denoted by $S=\{S(n), n\in D\}$. Let $a\in M$, if for each $P\in\eta(a)$, S is eventually not in P, then we say that a is a limit point of S (or S converges to a, in symbols $S\to a$); The union of all limit points of S will be denoted by $\lim S$. If for each $P\in\eta(a)$, S is frequently not in P, then we say that a is a cluster point of S and in symbols $S\infty a$. The union of all cluster points of S will be denoted by $\mathrm{ad}S$.

It is clear that every limit point of S is a cluster point of S but not vice versa. One readily checks that a is a limit point of A iff $a\leqslant \lim S$, and a is a cluster point of A iff $a\leqslant \mathrm{ad}S$.

Theorem 4 Let $(L(M),\ \eta)$ be TML, $A\in L$, $a\in M$, then $a\leqslant A^-$ iff there exists in A a molecular net S which converges to a.

Proof Let S be a molecular net in A and $S\to a$, then for every $P\in\eta(a)$ S is eventually not in P, hence there is $n\in D$ such that $S(n)\not\leqslant P$ and so $A\not\leqslant P$, i.e., $a\leqslant A^-$.

Conversely, if $a\leqslant A^-$, then $A\not\leqslant P$ holds for each $P\in\eta(a)$. Choose $S(P)\in M$ such that $S(P)\not\leqslant P$ and $S(P)\leqslant A$, then the molecular net $S=\{S(P),\ P\in\eta(a)\}$ is what we need.

§5 Generalization of C. T. Yang's Theorem

Definition 7 Let $L(M)$ be a molecular lattice, $A\in L$, $A\neq 0$, $m\in M$. m is called a component of A if (i) $m\leqslant A$, and (ii) $m'\in M$, $m'\geqslant m$ and $m'\leqslant A$, imply that $m'=m$. Components of 1 will be called maximal points.

Example (1) If $L=\mathscr{P}(X)$, $A\neq\varnothing$, then every point of A is a component of A. (2) If $L=I^X$, $A\in L$, $A\neq 0$, then a point x_λ is a component of A iff $A(x)=\lambda$.

In (1) and (2), if m_1 and m_2 are different components of A, then $m_1\wedge m_2=0$.

Theorem 5 Let L be a molecular lattice, $A\in L$, $A\neq 0$, $a\in M$ and $a\leqslant A$, then A has at least one component m such that $a\leqslant m$.

Proof Let φ be a chain in L. We say that φ is in A if $\forall x\in\varphi$, $x\leqslant A$, in symbol, $\varphi\leqslant A$. Consider the family of chains

$$\mathbf{C}=\{\varphi\,|\,a\in\varphi\subset M,\ \varphi\leqslant A\}.$$

Since $\{a\}\in\mathbf{C}$ we have $\mathbf{C}\neq\varnothing$. Assume that

$$\varphi_1\leqslant\varphi_2 \text{ iff } \varphi_1\subset\varphi_2,$$

then $\mathbf{C}$ becomes a poset. It is clear that every totally ordered subset of $\mathbf{C}$ has an upper bound, hence there exists a maximal element $\varphi_0\in\mathbf{C}$. Let $m=\sup\varphi_0$, then $a\leqslant m\leqslant A$. One readily checks that m is $\vee$-irreducible, i. e., $m\in M$, and m is a component of A such that $a\leqslant m$.

Remark The component mentioned in theorem 5 may be not uniqne. For example, let $L=\{0,\ 1,\ a,\ m,\ m'\}$, define $a\leqslant m$, $a\leqslant m'$ and m and m' are incomparable, then $L(M)$ is a molecular lattice where $M=\{a, m,\ m'\}$. Now $A=1$ has two different components m and m' such that both $a\leqslant m$ and $a\leqslant m'$ are true.

Proposition Let $L(M)$ be a molecular lattice, $A\in L$, then for each point

$a \leqslant A$, A has a unique component $m(a, A)$ such that $a \leqslant m(a, A)$ iff different components of A are disjoint, i. e., their intersections are equal to 0.

The proof is obvious.

The C. T. Yang theorem concerning derived element in general topology can be generalized to the case of TML.

Theorem 6 Let $(L(M), \eta)$ be a TML and $\forall A \in L$, different components of A are disjoint, then the derived element of every element is closed iff the derived element of every point is closed.

Proof We only consider the sufficiency. Suppose that $a \in M$, $a \leqslant (A^d)^-$, we have to prove that $a \leqslant A^d$. If $a \nleqslant A$, then, by virtue of the fact that $a \leqslant A^-$ we have $a \leqslant A^d$. Hence we may assume that $a \leqslant A$. Let $m = m(a, A)$ be the unique component of A such that $a \leqslant m$.

(i) $a \leqslant m^d$. Let $\beta^*(a)$ be the standard minimal family of a, then we need only to prove that $\forall x \in \beta^*(a)$, $x \leqslant A^d$. In fact, $\forall x \in \beta^*(a)$ since $m^d = \bigvee \{y \mid y$ is an accumulation point of m$\} \geqslant a$, the point m has an accumulation point d_x such that $d_x \geqslant x$. By the meaning of d_x one readily checks that $d_x \nleqslant m$, hence $d_x \nleqslant A$ (because otherwise there will be at least two components of A containning the same point x). On the other hand, $d_x \leqslant m^- \leqslant A^-$, hence d_x is an accumulation point of A and this proves that $x \leqslant d_x \leqslant A^d$.

(ii) $a \nleqslant m^d$. Since m^d is closed we have $m^d \in \eta(a)$. Suppose that $P \in \eta(a)$, let $P_1 = m^d \vee P$, then $P_1 \in \eta(a)$. Note that $a \leqslant (A^d)^-$, i. e., a is an adherence point of A^d, we have $A^d \nleqslant P_1$, hence A has an accumulation point c such that $c \nleqslant P_1$ and so $P_1 \in \eta(c)$. If $c \nleqslant m^-$, then $m^- \vee P_1 \in \eta(c)$ and hence $A \nleqslant m^- \vee P_1$; If $c \leqslant m^-$, then it follows by the fact $c \nleqslant P_1$ and $m^d \leqslant P_1$ that $c \nleqslant m^d$, hence $c \leqslant m \leqslant A$. Moreover, by the meaning of c and the fact that $c \leqslant A$ we know that $A \nleqslant m \vee P_1$. In both cases we have $A \nleqslant m \vee P$. This proves that a is an accumulation point of A, i. e., $a \leqslant A^d$.

§4 Connectivity

In general topology, there are different ways to describe connectivity of a subset, may be the K. Fan theorem is the most interesting one. In this section, we shall define connectivity of elements of a TML in this way.

For the sake of convenience, we agree to the appointment that for every TML L and $\forall A \in L$, $[A] = \{x \in M \mid x \leqslant A\}$.

Definition 8 Let $(L(M), \eta)$ be a TML, $A \in L$, $A \neq 0$. If for each pair a, b of points of $[A]$ and each mapping

$$P: [A] \to \bigcup \{\eta(x) \mid x \in [A]\}, \text{ where } \forall x \in [A],\ P_x = P(x) \in \eta(x),$$

there exists in $[A]$ a finite number of points $x_1 = a$, x_2, $\cdots$, $x_n = b$ such that

$$A \nleqslant P_{x_i} \vee P_{x_{i+1}}, \quad i=1, \cdots, n-1, \tag{C}$$

then A is said to be connected. Moreover, we assume that 0 is connected.

If A is a maximal connected element of L, then A is called a connected component.

Theorem 7 Suppose that A is connected and $A \leqslant B \leqslant A^-$, then B is connected.

Proof We only consider the case when $A \neq 0$. Suppose that $a, b \in [B]$ and

$$P: [B] \to \cup\{\eta(x) \mid x \in [B]\}, \text{ where } \forall x \in [B], \ P_x \in \eta(x)$$

is a mapping. Since $a \leqslant A^-$, we have $A \nleqslant P_a$, hence there exists $c \in [A]$ such that $c \nleqslant P_a$ and so $P_a \in \eta(c)$. Similarly, there exists $d \in [A]$ such that $P_b \in \eta(d)$. Consider the mapping

$$P': [A] \to \cup\{\eta(x) \mid x \in [A]\}$$

defined by

$$P'(x) = \begin{cases} P_x, & \text{if } x \neq c, \ x \neq d, \\ P_a, & \text{if } x = c, \\ P_b, & \text{if } x = d. \end{cases}$$

Since A is connected, there exists in $[A]$ points $x_1 = c, x_2, \cdots, x_n = d$ such that (C) holds. Hence there exist in $[B]$ points $x_1 = a, x_2, \cdots, x_n = b$ such that

$$B \nleqslant P_{x_i} \vee P_{x_{i+1}}, \quad i = 1, \cdots, n-1.$$

Therefore B is connected.

Lemma Suppose that $[A]_0 \subset [A]$ and $\vee\{x \mid x \in [A]_0\} = A$. If for each pair a, of points of $[A]_0$ and each mapping

$$P': [A_0] \to \cup\{\eta(x) \mid x \in [A]_0\}, \text{ where } \forall x \in [A]_0, \ P'_x \in \eta(x),$$

there exists in $[A]_0$ a finite number of points $x_1 = a, x_2, \cdots, x_n = b$ such that (C) holds, then A is connected.

Proof Let $a, b \in [A]$ and

$$P: [A] \to \cup\{\eta(x) \mid x \in [A]\}, \text{ where } \forall x \in [A], \ P_x \in \eta(x)$$

be a mapping. Simce $A \nleqslant P_a$, there exists $c \in [A]_0$ such that $P_a \in \eta(c)$. Similarly there exists $d \in [A]_0$ such that $P_b \in \eta(d)$. Define

$$P': [A]_0 \to \cup\{\eta(x) \mid x \in [A]_0\}$$

as follows

$$P'_x = \begin{cases} P_x, & \text{if } x \neq c, \ x \neq d, \\ P_a, & \text{if } x = c, \\ P_b, & \text{if } x = d. \end{cases}$$

By the condition given in this lemma there exists in $[A]_0$ points $x_1 = c, x_2, \cdots, x_n = d$ such that (C) holds. Instead of c and d we put $x_1 = a$ and $x_n = b$ respectively and keep $x_2, \cdots, x_{n-1}$ unchanged, then (C) is still ture and hence A is connected.

Let us agree to call A and B separated in case $A^- \wedge B = A \wedge B^- = 0$.

Theorem 8 Suppose that A and B are connected elements of $L(M)$ and they are not separated, then $A \vee B$ is connected.

Proof Without loss of generality we may assume that $A^- \vee B \neq 0$. Put $[A \vee B]_0 = [A] \vee [B]$, then $\vee [A \vee B]_0 = A \vee B$. Suppose that $a, b \in [A \vee B]_0$ and

P: $[A \vee B]_0 \to \cup \{\eta(x) \mid x \in [A \vee B]_0\}$, where $\forall x \in [A \vee B]_0$, $P_x \in \eta(x)$

is a mapping. In cases $a, b \in [A]$ or $a, b \in [B]$, the proof is easy, we need only to consider the case when $a \in [A]$, $b \in [B]$. Choose a point $d \leqslant A^- \wedge B$, then $d \leqslant A^-$, and $d \leqslant B$. Since $d \leqslant A^-$, $A \not\leqslant P_d$, hence there exist $c \in [A]$ such that $c \not\leqslant P_d$ and so $P_d \in \eta(c)$. Since A is connected, there exists in $[A]$ points $x_1 = a$, $x_2, \cdots, x_n = c$ such that (C) holds. Similarly, since B is connected, there exists in $[B]$ points $y_1 = d, y_2, \cdots, y_m = b$ such that

$$B \not\leqslant P_{y_i} \vee P_{y_{i+1}}, \quad i = 1, \cdots, m-1.$$

Put

$$z_i = \begin{cases} x_i, & i \leqslant n, \\ y_{i-n}, & n < i \leqslant m+n, \end{cases} \qquad i = 1, \cdots, m+n-1,$$

then $\forall i$, $z_i \in [A \vee B]_0$ and since $c \not\leqslant P_c \vee P_d$, we have $A \vee B \not\leqslant P_c \vee P_d = P_{z_n} \vee P_{z_{n+1}}$. Hence

$$A \vee B \not\leqslant P_{z_i} \vee P_{z_{i+1}}, \quad i = 1, \cdots, m+n-1,$$

and by the lemma we know that $A \vee B$ is connected.

In a similar way we can use the lemma to prove the following more general result.

Theorem 9 Let $(L(M), \eta)$ be a TML, $\{A_i \mid i \in I\}$ be a set of connected elements of $(L(M), \eta)$, and there exists $i_0 \in I$ such that $\forall i \in I$, A_i and A_{i_0} are not separated, then $\vee \{A_i \mid i \in I\}$ is connected.

By virtue of theorem 6 and theorem 9 and the fact that every molecule is connected we get the following theorem.

Theorem 10 Let $(L(M), \eta)$ be a TML, then

(1) $\forall x \in m$, there exists a connected component A_x such that $x \leqslant A_x$,

(2) every connected component is closed,

(3) different connected components are separated.

Hence the greatest element 1 can be decomposed into disjoint closed maximal connected elements.

Definition 9 Let $L_1(M_1)$, $L_2(M_2)$ be molecular lattices, a mapping f: $L_1 \to L_2$ is called a generlized order-homomorphism, or briefly, a GOH, if

(1) $f(a) = 0$ iff $a = 0$,

(2) f is union-preserving,

(3) f^{-1} is union-preserving, where $\forall b \in L_2$

$$f^{-1}(b)=\bigvee\{a\in L_1\,|\,f(a)\leqslant b\}.$$

The concept of GOH is a generlization of the concepts of mappings, fuzzy mappings and the concept of order-homomorphisms, the last one was introduced in [3] and was thoroughly investigated in [6].

It is not difficult to verify that a GOH f maps molecules into molecules, i. e., $f(M_1)\subset M_2$. Moreover, f^{-1}: $L_2\to L_1$ is intersection-preserving.

Theorem 11 Suppose that f: $L_1(M_1)\to L_2(M_2)$ is a continuous GOH and A is connected in L_1, then $f(A)$ is connected in L_2.

Proof We only consider the case when $A\neq 0$. Let

$$[f(A)]_0=\{f(x)\,|\,x\in[A]\},$$

then $[f(A)]_0\subset[f(A)]$ and

$$\bigvee[f(A)]_0=\bigvee\{f(x)\,|\,x\in[A]\}=f(\bigvee[A])=f(A).$$

Suppose that $f(A)$ is not connected, then by the lemma there exist a mapping

$$Q:\ [f(A)]_0\to\bigcup\{\eta(y)\,|\,y\in[f(A)]_0\},\ \text{where}\ \forall y\in[f(A)]_0,\ Q_y\in\eta(y),$$

and two points $a,b\in[A]$ such that for any finite number of points $x_1=a,\ x_2,\cdots,x_n=b$ of $[A]$, there exists $i\leqslant n-1$ such that

$$f(A)\leqslant Q_{f(x_i)}\vee Q_{f(x_{i+1})},\quad i=1,\cdots,n-1.$$

$\forall x\in[A]$, let $P_x=f^{-1}(Q_{f(x)})$, then P_x is closed and $x\not\leqslant P_x$ because of $f(x)\not\leqslant Q_{f(x)}$. Hence $P_x\in\eta(x)$ and we get a mapping

$$P:\ [A]\to\bigcup\{\eta(x)\,|\,x\in[A]\},\ \text{where}\ \forall x\in[A],\ P_x\in\eta(x).$$

Consider the points a and b. From the discussion mentioned above we know that for any finite number of points $x_1=a,\ x_2,\cdots,\ x_n=b$ of $[A]$ there exists some $i\leqslant n-1$ such that

$$A\leqslant f^{-1}f(A)\leqslant f^{-1}(Q_{f(x_i)}\vee Q_{f(x_{i+1})})=f^{-1}(Q_{f(x_i)})\vee f^{-1}(Q_{f(x_{i+1})})=P_{x_i}\vee P_{x_{i+1}}.$$

This shows that A is not connected.

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