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An Example of the 2-Uniformly Rotund Banach Spaces*

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The conception of the k-UR (k-uniformly rotund) Banach spaces was given by F. Sullivan in 1979[1]. He pointed out that there exist some Banach spaces which are not $(k-1)$-UR but k-UR. It is easily to show such spaces if they are not rotund.

In this note an example is given in which the Banach space is not 1-UR (i.e., UR) but rotund and 2-UR.

In[4] we gave an equivalent form of the k-uniform rotundity:

A real Banach space X is 2-UR if and only if for any sequences $\{x^{(m)}\}$, $\{y^{(m)}\}$ and $\{z^{(m)}\}$ of X, if $\|x^{(m)}\|\to 1$, $\|y^{(m)}\|\to 1$, $\|z^{(m)}\|\to 1$ and $\|x^{(m)}+y^{(m)}+z^{(m)}\|\to 3$ as $m\to\infty$, then

$$\sup\left\{\left|\begin{matrix}1 & 1 & 1\\ f(x^{(m)}) & f(y^{(m)}) & f(z^{(m)})\\ g(x^{(m)}) & g(y^{(m)}) & g(z^{(m)})\end{matrix}\right| : f,g\in B(X^*)\right\}\to 0$$

as $m\to\infty$, where $B(X^*)$ is the dual unit ball.

Let $X=l^2$, $x=(x_1,x_2,\cdots,x_n\cdots)\in X$, $B(X)$ be the convex body defined by

$$B(X)=\left\{x\in X:\ \text{if } \frac{\sqrt{7}}{4}\leqslant|x_1|\leqslant 1,\ \text{then} \sum_{n=1}^{\infty}x_n^2\leqslant 1 \text{ and}\right.$$
$$\left.\text{if } |x_1|\leqslant\frac{\sqrt{7}}{4},\ \text{then} \sum_{n=2}^{\infty}\frac{x_n^2}{\left(\sqrt{n^2-\frac{1}{2}n+\frac{1}{2}-x_1^2}-(n-1)\right)^2}\leqslant 1\right\}.$$

The Minkowsky functional of $B(X)$ is refered to as a new norm $\|\cdot\|_*$ of X, i.e., $B(X)$ is the unit ball in $(X,\|\cdot\|_*)$ (The origin norm of X is denoted by $\|\cdot\|_2$).

Now we prove that the Banach space $(X,\|\cdot\|_*)$ is not UR but R and 2-UR.

Lemma 1 Suppose $x=(x_1,x_2,\cdots,x_n,\cdots)\in B(X)$, $y=(y_1,y_2,\cdots,y_n,\cdots)\in B(X)$,

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$|x_1|\leqslant\frac{\sqrt{7}}{4}$, $|y_1|\leqslant\frac{\sqrt{7}}{4}$ and

$$f_n(t)=\frac{(tx_n+(1-t)y_n)^2}{\left(\sqrt{n^2-\frac{1}{2}n+\frac{1}{2}-(tx_1+(1-t)y_1)^2}-(n-1)\right)^2},\quad 0\leqslant t\leqslant 1,\ n=2,3,\cdots \tag{1}$$

If neither $x_1-y_1=x_n-y_n=0$ nor $x_n=y_n=0$, then $f_n(t)$ is a strictly convex function in the unit interval.

Proof For $0\leqslant t\leqslant 1$, $|tx_1+(1-t)y_1|\leqslant\frac{\sqrt{7}}{4}$, and so

$$\sqrt{n^2-\frac{1}{2}n+\frac{1}{2}-(tx_1+(1-t)y_1)^2}\geqslant n-\frac{1}{4}. \tag{2}$$

Differentiating $f_n(t)$ twice and noting (by(2))

$$\left(n^2-\frac{1}{2}n+\frac{1}{2}-(tx_1+(1-t)y_1)^2\right)^{3/2}-(n-1)\left(n^2-\frac{1}{2}n+\frac{1}{2}\right)>1.$$

We can get $f_n''(t)>0$ if neither $x_1-y_1=x_n-y_n=0$ nor $x_n=y_n=0$.

Proposition 1 $(X,\|\cdot\|_*)$ is rotund.

Proof Suppose $x\neq y$, $\|x\|_*=\|y\|_*=1$. What we need to prove is $\left\|\frac{x+y}{2}\right\|_*<1$. It can be proved in three cases:

(i) $|x_1|\leqslant\frac{\sqrt{7}}{4}$ and $|y_1|\leqslant\frac{\sqrt{7}}{4}$; (ii) $|x_1|\geqslant\frac{\sqrt{7}}{4}$ and $|y_1|\geqslant\frac{\sqrt{7}}{4}$;

(iii) $|x_1|>\frac{\sqrt{7}}{4}$ and $|y_1|<\frac{\sqrt{7}}{4}$ or $|x_1|<\frac{\sqrt{7}}{4}$ and $|y_1|>\frac{\sqrt{7}}{4}$.

It is trivial to prove $\left\|\frac{x+y}{2}\right\|_*<1$ in the cases (i) and (ii). For the case (iii), without loss of generality, we can assume

$$x_1>\frac{\sqrt{7}}{4}\ \text{and}\ -\frac{\sqrt{7}}{4}<y_1<\frac{\sqrt{7}}{4}.\ \text{Let}\ \lambda=\frac{\frac{\sqrt{7}}{4}-y_1}{x_1-y_1},$$

$$\beta^2=\sum_{n=2}^{\infty}(\lambda x_n+(1-\lambda)y_n)^2,\ z_1=\frac{\sqrt{7}}{4},\ z_n=\frac{3}{4\beta}(\lambda x_n+(1-\lambda)y_n)\ (n=2,3,\cdots),$$

$z=(z_1,z_2,\cdots,z_n,\cdots)$ and $w=\lambda x+(1-\lambda)y$.

Then

$$\|w\|_*<\|z\|_*=1.$$

For $0<t<1$, we have

$$\|tw+(1-t)x\|_*=\|tw+(1-t)x\|_2\leqslant t\|w\|_2+(1-t)\|x\|_2<1 \tag{3}$$

and (by lemma 1)

$$\|tw+(1-t)y\|_*<\sum_{n=2}^{\infty}\frac{(t|z_n|+(1-t)y_n)^2}{\left(\sqrt{n^2-\frac{1}{2}n+\frac{1}{2}-(tz_1+(1-t)y_1)^2}-(n-1)\right)^2}<1. \tag{4}$$

From (3) and (4), it is not difficult to prove $\left\| \frac{x+y}{2} \right\|_* < 1$.

Proposition 2 $(X, \|\cdot\|_*)$ is not uniformly rotund. In fact, $(X, \|\cdot\|_*)$ is not URED (URED = uniformly rotund in every direction)[a].

Proof Let $\{x^{(m)};\ m=2,3,\cdots\}$ and $\{y^{(m)};\ m=2,3,\cdots\}$ be the two sequences of unit vectors in $(X, \|\cdot\|_*)$, where $x^{(m)}=(x_1^{(m)}, x_2^{(m)}, \cdots, x_n^{(m)}\cdots)$, $y^{(m)}=(y_1^{(m)}, y_2^{(m)}, \cdots y_n^{(m)}, \cdots)$ $(m=2,3,\cdots)$ and

$$x_n^{(m)}=\begin{cases} \frac{\sqrt{7}}{4} & \text{when } n=1, \\ \frac{4}{3} & \text{when } n=m, \\ 0 & \text{or else;} \end{cases} \qquad y_n^{(m)}=\begin{cases} -\frac{\sqrt{7}}{4} & \text{when } n=1, \\ \frac{3}{4} & \text{when } n=m, \\ 0 & \text{or else.} \end{cases}$$

Thus,

$$\left\| \frac{x^{(m)}+y^{(m)}}{2} \right\|_* = \frac{\frac{3}{4}}{\sqrt{m^2-\frac{1}{2}m+\frac{1}{2}}-(m-1)} \to 1 \text{ as } m \to \infty,$$

and

$$x^{(m)}-y^{(m)}=\frac{\sqrt{7}}{2}e_1,$$

where $e_1=(1,0,,\cdots,0,\cdots)$. Therefore $(X, \|\cdot\|_*)$ is not URED and so is not UR.

Lemma 2 Let N be a fixed natural number, $x, y \in B(X)$, $|x_1| \leqslant \frac{\sqrt{7}}{4}$ and $|y_1| \leqslant \frac{\sqrt{7}}{4}$. For every $\delta_1>0$, there is $\delta_2(\delta_1, N)>0$ such that if $|x_n-y_n|>2\delta_1$ for some $n \leqslant N$

$$\frac{1}{2}(f_n(0)+f_n(1))-f_n\left(\frac{1}{2}\right) \geqslant \delta_2,$$

where $f_n(t)$ is defined in (1).

Proof The lemma can be proved in two cases.

(i) $\left|\frac{x_n+y_n}{2}\right| \leqslant \min(|x_n|, |y_n|)$.

By lemma 1, we can get

$$\frac{1}{2}(f_n(0)+f_n(1))-f_n\left(\frac{1}{2}\right) \geqslant \frac{\frac{1}{2}\left(x_n^2-\left(\frac{x_n+y_n}{2}\right)^2\right)}{\left(\sqrt{n^2-\frac{1}{2}n+\frac{1}{2}-x_1^2}-(n-1)\right)^2}+$$

$$+\frac{\frac{1}{2}\left(y_n^2-\left(\frac{x_n+y_n}{2}\right)^2\right)}{\left(\sqrt{n^2-\frac{1}{2}n+\frac{1}{2}-y_1^2}-(n-1)\right)^2}.$$

If $|x_n|\geqslant|y_n|$, then $|x_n|-\left|\frac{x_n+y_n}{2}\right|=\left|\frac{x_n-y_n}{2}\right|>\delta_1$ and $|x_n|^2-\left|\frac{x_n+y_n}{2}\right|^2>\delta_1^2$; and if $|y_n|>|x_n|$ then $|y_n|^2-\left|\frac{x_n+y_n}{2}\right|^2>\delta_1^2$. Hence

$$\frac{1}{2}(f_n(0)+f_n(1))-f_n\left(\frac{1}{2}\right)>\frac{1}{2}\delta_1^2.$$

(ii) $\left|\frac{x_n+y_n}{2}\right|>\min(|x_n|,|y_n|)$.

Without loss of generality, $\left|\frac{x_n+y_n}{2}\right|>|y_n|$, and so $\left|\frac{x_n+y_n}{2}\right|>\frac{\delta_1}{2}$.

For the given δ_1, there is $\delta'(\delta_1,N)>0$ such that if $|x_1-y_1|<\delta'$ and $n\leqslant N$

$$\left|\frac{1}{\left(\sqrt{n^2-\frac{1}{2}n+\frac{1}{2}-x_1^2}-(n-1)\right)^2}-\frac{1}{\left(\sqrt{n^2-\frac{1}{2}n+\frac{1}{2}-y_1^2}-(n-1)\right)^2}\right|<\frac{\delta_1^2}{2}.$$

Thus, for $|x_1-y_1|<\delta'$,

$$\frac{1}{2}(f_n(0)+f_n(1))-f_n\left(\frac{1}{2}\right)\geqslant\left(\frac{x_n-y_n}{2}\right)^2-\frac{1}{2}\delta_1^2>\frac{1}{2}\delta_1^2.$$

If $|x_1-y_1|\geqslant\delta'$, by lemma 1 for $t\in\left[\frac{3}{8},\ \frac{5}{8}\right]$, we have

$$|tx_n+(1-t)y_n|>\frac{1}{4}\delta_1,$$

and so

$$f_n''(t)\geqslant\frac{2\left(\frac{1}{4}\delta_1\right)^2\cdot\delta'^2}{\left(\sqrt{N^2-\frac{1}{2}N+\frac{1}{2}}\right)^3}=\delta_3>0.$$

Therefore

$$\frac{1}{2}(f_n(0)+f_n(1))-f_n\left(\frac{1}{2}\right)\geqslant\frac{1}{2}\left(f_n\left(\frac{3}{8}\right)+f_n\left(\frac{5}{8}\right)\right)-f_n\left(\frac{1}{2}\right)>\frac{\delta_3}{128}>0.$$

Taking $\delta_2=\min\left(\frac{1}{2}\delta_1^2,\frac{\delta_3}{128}\right)$, the proof is finished.

Lemma 3 Suppose $\{x^{(m)}\}$ and $\{y^{(m)}\}$ are two sequences of unit vectors in $(X,\|\cdot\|_*)$ where $x^{(m)}=(x_1^{(m)},x_2^{(m)},\cdots,x_n^{(m)},\cdots)$ and $y^{(m)}=(y_1^{(m)},y_2^{(m)},\cdots,y_n^{(m)},\cdots)$ $(m=1,2,\cdots)$. If $\left\|\frac{x^{(m)}+y^{(m)}}{2}\right\|_*\to1$ as $m\to\infty$, then $\sum_{n=2}^{\infty}(x_n^{(m)}-y_n^{(m)})^2\to0$ as $m\to\infty$.

Proof If the conclusion of the lemma is false, there exist subsequences of $\{x^{(m)}\}$ and $\{y^{(m)}\}$ (we still denote them by $\{x^{(m)}\}$ and $\{y^{(m)}\}$) such that

$$\sum_{n=2}^{\infty}\left(\frac{x_n^{(m)}-y_n^{(m)}}{2}\right)^2\geqslant\delta>0. \tag{5}$$

(i) For all m, $|x_1^{(m)}|\leqslant\frac{\sqrt{7}}{4}$, $|y_1^{(m)}|\leqslant\frac{\sqrt{7}}{4}$.

For the given δ in (5) there is a natural number N such that if $n>N$ and $|\alpha|\leqslant\frac{\sqrt{7}}{4}$.

$$\frac{16}{9}-\frac{\delta}{8}\leqslant\frac{1}{\left(\sqrt{n^2-\frac{1}{2}n+\frac{1}{2}-\alpha^2}-(n-1)\right)^2}\leqslant\frac{16}{9}.$$

On the other hand, according to the assumption of the lemma, there is M such that if $m>M$, $\left\|\frac{x^{(m)}+y^{(m)}}{2}\right\|_*>1-\frac{\delta}{8}$. Hence by lemma 1,

$$\sum_{n=2}^{N}\left(\frac{x_n^{(m)}-y_n^{(m)}}{2}\right)^2>\delta-\sum_{n=N+1}^{\infty}\left(\frac{x_n^{(m)}-y_n^{(m)}}{2}\right)^2$$

$$\geqslant\delta-\frac{9}{16}\Bigg[\sum_{n=N+1}^{\infty}\Bigg(\frac{\frac{1}{2}x_n^{(m)2}}{\left(\sqrt{n^2-\frac{1}{2}n+\frac{1}{2}-y_1^{(m)2}}-(n-1)\right)^2}+$$

$$+\frac{\frac{1}{2}y_n^{(m)2}}{\left(\sqrt{n^2-\frac{1}{2}n+\frac{1}{2}-y_1^{(m)2}}-(n-1)\right)^2}$$

$$-\frac{\left(\frac{x_n^{(m)}+y_n^{(m)}}{2}\right)^2}{\left(\sqrt{n^2-\frac{1}{2}n+\frac{1}{2}-\left(\frac{x_1^{(m)}+y_1^{(m)}}{2}\right)^2}-(n-1)\right)^2}\Bigg)+\frac{\delta}{8}\Bigg]$$

$$\geqslant\delta-\frac{9}{16}\left[\frac{1}{2}(\|x^{(m)}\|_*^2+\|y^{(m)}\|_*^2)-\left\|\frac{x^{(m)}+y^{(m)}}{2}\right\|_*^2+\frac{\delta}{8}\right]>\frac{55}{64}\delta.$$

For the given δ in (5), N is a fixed number, so there is n, $2\leqslant n\leqslant N$ such that for infinite number of m, $m>M$.

$$\left|\frac{x_n^{(m)}-y_n^{(m)}}{2}\right|^2>\frac{55\delta}{64N}=\delta_1^2>\theta.$$

By lemma 2, there is $\delta_2(\delta_1,N)>0$ such that $\left\|\frac{x^{(m)}+y^{(m)}}{2}\right\|_*^2\leqslant1-\delta_2$ which contradicts the assumption $\left\|\frac{x^{(m)}-y^{(m)}}{2}\right\|_*\to1$ as $m\to\infty$.

(ii) There is infinite number of m such that $|x_1^{(m)}|\leqslant\frac{\sqrt{7}}{4}$ and $|y_1^{(m)}|\leqslant\frac{\sqrt{7}}{4}$. Choosing subsequences of $\{x^{(m)}\}$ and $\{y^{(m)}\}$, we can get the contradiction in the same way.

If for infinite number of m, $|x_1^{(m)}|\geqslant\frac{\sqrt{7}}{4}$, $|y_1^{(m)}|\geqslant\frac{\sqrt{7}}{4}$ we can get the same contradiction as usual.

(iii) For almost all m, $|x_1^{(m)}|\leqslant\frac{\sqrt{7}}{4}$ and $|y_1^{(m)}|>\frac{\sqrt{7}}{4}$ or $|x_1^{(m)}|\geqslant\frac{\sqrt{7}}{4}$ and

$|y_1^{(m)}|<\frac{\sqrt{7}}{4}$. For these m, let $\lambda_m=\frac{\frac{\sqrt{7}}{4}-y_1^{(m)}}{x_1^{(m)}-y_1^{(m)}}$ and $z^{(m)}=\lambda_m x^{(m)}+(1-\lambda_m)y^{(m)}$.

It is readily to check that $\|z^{(m)}\|_*\to 1$, $\left\|\frac{x^{(m)}+z^{(m)}}{2}\right\|_*\to 1$ and $\left\|\frac{z^{(m)}+y^{(m)}}{2}\right\|_*\to 1$ as $m\to\infty$. According to the proof in(ii), $\sum_{n=2}^{\infty}(x_n^{(m)}-z_n^{(m)})^2\to 0$ and $\sum_{n=2}^{\infty}(z_n^{(m)}-y_n^{(m)})^2\to 0$ as $m\to\infty$. Hence $\sum_{n=2}^{\infty}(x_n^{(m)}-y_n^{(m)})^2\to 0$ as $m\to\infty$.

Proposition 3 $(X,\|\cdot\|_*)$ is 2-uniformly rotund.

Proof Suppose $\{x^{(m)}\}$, $\{y^{(m)}\}$ and $\{z^{(m)}\}$ are the sequences of unit vectors in $(X,\|\cdot\|_*)$ and $\|x^{(m)}+y^{(m)}+z^{(m)}\|_*\to 3$ as $m\to\infty$. By lemma 3,5 in [4], what we need to prove is

$$\sup\left\{\left|\begin{matrix}1 & 1 & 1\\ f(x^{(m)}) & f(y^{(m)}) & f(z^{(m)})\\ g(x^{(m)}) & g(y^{(m)}) & g(z^{(m)})\end{matrix}\right| : f,g\in B(x^*)\right\}\to 0 \text{ as } m\to\infty. \qquad (6)$$

By lemma 3, $\sum_{n=2}^{\infty}(x_n^{(m)}-y_n^{(m)})^2\to 0$ and $\sum_{n=2}^{\infty}(x_n^{(m)}-z_n^{(m)})^2\to 0$ as $m\to\infty$, together with the properties of determinant, we can get (6).

Remark According to this way, some examples in which the Banach spaces are not $(k-1)$-UR but R and k-UR can be constructed;

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