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On the Blowing-up and Quenching Problems for Semilinear Heat Equation*

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1 Introduction

In recent years due to investigations on some practical problems it is found solutions of initial or initial-boundary value problems for semilinear heat equation and other equations may blow up or quench. See, for example, [1]-[9]. In [1] S. Kaplan discussed general parabolic equation

$$u_t = Lu + f(u),$$

where L is a linear uniformly elliptic operator of second order and proved: if

$$\int_t^{\infty} \frac{du}{f(u)} < \infty,$$

then solution u of initial or initial-boundary value problems for the above equation blow up, i.e., as t tends to some finite value $T^*>0$, $|u|$ tends to infinite. On the other hand H. Kawarada [5] had found: solution of the following problem

$$\begin{cases} u_t = u_{xx} + \dfrac{1}{1-u}, \quad 0<t<T, \quad |x|<a, \\ u(0,x) = 0, \quad u(t, \pm a) = 0 \end{cases} \tag{0}$$

may quench, i.e. there exists a critical value a^*, for which in case $a>a^*$, the derivations of u blow up.

In this article we consider the problem of preventing the occurrence of these phenomena. By introducing a sufficiently strong antibloeing-up factor $g(t)$ or anti-quenching factor $g(t,x)$ to the nonlinear term $f(u)$, we can preclude the blowing-up or quenching of the solution.

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2 The problem of anti-blowing-up

In order to give the essense without involving superfluous estimates, we consider the typical simple case:

$$\begin{cases} u_t = \Delta u + g(t) f(u), & t>0, \quad x\in R^n, \\ u|_{t=0} = u_0(x), & |u_0(x)|\leq M. \end{cases} \tag{1}$$

suppose $g(t)$ and $f(u)$ are smooth; moreover

$$A|u|^\alpha \leq |f(u)| \leq B|u|^\alpha, \quad A>0, \quad \alpha>1, \tag{2}$$

$$\int_0^\infty |g(t)|\,dt = K<1. \tag{3}$$

We use elementary iteration method. Denote

$$E(t,x) = (4\pi t)^{-\frac{n}{2}} exp\frac{-x^2}{4t}.$$

It is known $E\geq 0$ and

$$\int_{R^n} E(t,x-\xi)\,d\xi = \pi^{-\frac{n}{2}}\int_{R^n} e^{-\eta^2}\,d\eta = 1, \tag{4}$$

where $\eta = 2\sqrt{t}\,(x-\xi)$ means $\eta_j = 2\sqrt{t}\,(x_j-\xi_j)$, $j=1,\cdots,n$. Try a solution of the problem (1) by the following iterated sequence:

$$u_{j+1} = \int E(t,x-\xi)u_0(\xi)\,d\xi + \int_0^t\int E(t-\tau,x-\xi) f(u_j(\tau,\xi))g(\tau)\,d\xi d\tau, \quad j=0,1,2,\cdots,$$

hereafter we omit R^n in space integral. It is easy to get the elementary estimates:

$$|u_1| \leq M\int E(t,x-\xi)\,d\xi + M^\alpha B\int_0^\infty |g(\tau)|\,d\tau \leq M + BM^\alpha K,$$

$$|u_2| \leq M + \int_0^t |g(\tau)|\,d\tau\int EB|u_1|^\alpha d\xi \leq M + B(M+BM^\alpha K)^\alpha K,$$

in general

$$|u_{j+1}| \leq M + B\{M + [M + (M+\cdots)^\alpha K]^\alpha K\}^\alpha K,$$

where brackets occur j times. By the way we make a rather strong requirement:

$$M + BK = \varepsilon < 1, \tag{6}$$

i.e., the initial data u_0 is sufficiently "small" and the antiblowing-up factor $g(t)$ is sufficiently "strong". Then it is seen

$$|u_1| \leq M + BM^\alpha K \leq M + BK = \varepsilon, \quad |u_2| \leq M + B(M+BM^\alpha K)^\alpha K \leq M + B\varepsilon^\alpha K \leq M + BK = \varepsilon,$$

in general

$$|u_{j+1}| \leq \varepsilon, \quad j = 0,1,2,\cdots, \tag{7}$$

This gives the uniform boundedness of the iterated sequence $\{u_j\}$.

In order to affirm that this sequence has a uniform limit, we shall prove the uniform convergence of the series

$$u_0 + (u_1-u_0) + (u_2-u_1) + \cdots.$$

Since

$$|u_2 - u_1| \leq \int_0^t |g(\tau)|\,d\tau\int E|f(u_1) - f(u_0)|\,d\xi \leq 2B\varepsilon^\alpha K,$$

$$|u_3 - u_2| \leqslant \int_0^t |g(\tau)|\, d\tau \int_E |f(u_2) - f(u_1)|\, d\xi \leqslant K \sup \left|\frac{df}{du}\right| |u_2 - u_1|,$$

we suppose

$$\left|\frac{df(u)}{du}\right| \leqslant C|u|^{\beta}, \quad \beta > 0. \tag{8}$$

Then we have

$$|u_3 - u_2| \leqslant C\varepsilon^{\beta} 2B\varepsilon^{\alpha} K^2 = 2BC\varepsilon^{\alpha+\beta} K^2,$$

in general

$$|u_{j+1} - u_j| \leqslant 2BC^{j-1}\varepsilon^{\alpha+\beta j} K^j.$$

One can require K is sufficiently small such that

$$\sum_j C^{j-1} \varepsilon^{\alpha+\beta j} K^j$$

converges. This proves the uniform convergence of the sequence $\{u_j\}$. Its limit function is obviously the global solution of the problem (1) and does not blowup.

Thus we get the following result.

Theorem 1 Let the nonlinear term $f(u)$ possess the properties (2) and (8). Suppose the initial function $u_0(x)$ and the factor $g(t)$ are such that (6) holds, and

$$C\varepsilon K < 1. \tag{9}$$

Then for the problem (1) there exists a global solution for all $t > 0$ and no blowing-up occurs.

Remark 1 The central thesis of this paper is the introduction of the anti-blowing-up factor and is not the optimum of results. Hence the assumption of some superficially strong requirement perhaps does not lessen the meaning of this paper.

Example 1 It is easily verified, the following problem

$$u_t = u_{xx} + u^2, \quad u(0, x) = 1$$

has the solution $u = \frac{1}{1-t}$, which blows up as $t \to 1$. But the problem with anti-blowing-up factor $g(t) = (1+t)^{-3}$

$$u_t = u_{xx} + \frac{u^2}{(1+t)^3}, \quad u(0, x) = 1$$

has the solution $u = \frac{2(1+t)^2}{1+(1+t)^2}$, which exists globaly for all $t > 0$ and does not blow up.

3 The problem of anti-quenching

Similar discussion may by applied to the quenching phenomena. Consider on bounded domain $D \subset R^n$ the following problem:

$$\begin{cases} u_t = \Delta u + g(t, x) f(u), \quad t > 0, \quad x \in D; \\ u|_{t=0} = 0; \quad u|_{\partial D} = 0. \end{cases} \tag{10}$$

where $f(u)$ has singularity at $|u|=A>0$, but

$$\begin{cases} |f(u)|\leqslant M, & |u|\leqslant r<A, \quad f(0)\neq 0; \\ \left|\dfrac{df}{du}\right|\leqslant C, & |u|\leqslant r. \end{cases} \tag{11}$$

Suppose the anti-quenching factor $g(t,x)$ has the property:

$$\int_0^\infty\int_D |G(t-\tau,x,\xi)g(\tau,\xi)|d\xi d\tau\leqslant K<1 \tag{12}$$

where G is the Green's function of the corresponding linear homogeneous problem of (10). G exists for rather general domains D. Try a solution of (10) by iterated sequence:

$$u_{j+1}=\int_0^t\int_D G(t-\tau,x,\xi)g(t,\xi)f(u_j(\tau,\xi))d\xi d\tau,\quad j=0,1,\cdots,u_0(t,x)=0. \tag{13}$$

Note that problem (10) has the unique trivial solution $u\equiv 0$, when $f(0)=0$. Therefore in general we suppose $f(0)\neq 0$. In this case we can take $u_0\equiv 0$.

As in section 2, it is easily to obtain the following estimates;

$$|u_1|\leqslant\int_0^\infty\int_D |Gg|Md\xi d\tau\leqslant KM,\quad |u_2|\leqslant\int_0^\infty\int_D |Ggf(u_1)|d\xi d\tau\leqslant KM$$

and in general

$$|u_j|\leqslant KM,\quad j=1,2,\cdots.$$

Here we suppose the anti-quenching factor g is so strong that

$$KM\leqslant r. \tag{14}$$

Hence we always have $|f(u_j)|\leqslant M$, $j=1,2,\cdots$. This gives the uniform boundedness of the sequence $\{u_j\}$. Now we turn to prove the uniform convergence of the series

$$u_0+(u_1-u_0)+(u_2-u_1)+\cdots.$$

For this end we establish the following estimates:

$$|u_2-u_1|\leqslant\int_0^\infty\int_D |Gg|\,|f(u_1)-f(u_0)|d\xi d\tau\leqslant K|u_1-u_2|\sup\left|\frac{df}{dt}\right|\leqslant KCM,$$

$$|u_3-u_2|\leqslant\int_0^\infty\int_D |Gg|\,|f(u_2)-f(u_1)|d\xi d\tau\leqslant KC(KCM)=MK^2C^2,$$

in general

$$|u_{j+1}-u_j|\leqslant MK^jC^j,\quad j=1,2,\cdots.$$

Let g be so strong that

$$KC\leqslant a<1, \tag{15}$$

then the uniform convergence of the above mentioned sequence is confirmed. The limit function $u(t,x)$ of this sequence is defined for all $t>0$ and is the global solution of the problem (10). This solution does not quench. In fact, owing to the smoothness of the Green's function G for $t>0$, by the same discussion as above after differentiating both sides of (13), we can get the boundedness of derivatives of u.

Summarizing the above result, we get

Theorem 2 Suppose the nonlinear term $f(u)$ has the properties (11), and is singular at $|u|=A>0$. Let the factor $g(t,x)$ be such that (12), (14) , and (15) hold. Then the solution $u(t,x)$ of the problem (10) exists globally for all $t>0$ and does not quench.

Remark 2 In [5] Kawarada get an estimate for the critical value a^* of the problem (0):

$$a^*\leqslant\sqrt{2}.$$

In [6] a finer result is obtained:

$$0.765<a^*<\frac{\pi}{4}.$$

Here we can get a rather good lower bound by the elementary method used in this article. From the classical discussion we know the Green's function of the corresponding linear homogeneous problem of (0) is

$$G(t,x,\xi)=\frac{1}{a}\sum_{n=0}^{\infty}\exp\frac{-(2n+1)^2\pi^2}{4a^2}t\cos\frac{2n+1}{2a}x\cos\frac{2n+1}{2a}\xi.$$

Note that in (0) $g\equiv 1$, $f(0)=1$. We require $|u|\leqslant r<1$. It is easy to obtain the estimate

$$\int_0^t\int_{-a}^{a}|Gg|\,d\xi d\tau\leqslant\frac{1}{a}\int_0^t\int_{-a}^{a}\exp\frac{-(2n+1)^2\pi^2}{4a^2}(t-\tau)\,d\tau$$

$$\leqslant\frac{4a}{\pi^2}\sum\left[1-\exp\left(-\frac{(2n+1)^2\pi^2}{4a^2}t\right)\right]\frac{2a}{(2n+1)^2}\leqslant\frac{16a^2}{\pi^2}\sum\frac{1}{(2n+1)^2}=\frac{16a^2}{\pi^2}\frac{\pi^2}{8}=2a^2$$

for all $t>0$. Here we use the well-known result:

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8}.$$

In order that (12) holds, we must have

$$2a^2<1.$$

From this we get a lower bound of a^*

$$a^*\geqslant\frac{1}{\sqrt{2}}=0.707.$$

Though this result is not so fine, the calculation is quite simple.

Example 2 The following initial value problem

$$u_t=u_{xx}+\frac{1}{1-u},\quad u(0,x)=0$$

has the solution $u=1-\sqrt{1-2t}$, which quenches, i.e., the function stops taking real value as $t>1/2$ and its derivatives become infinite as $t\to 1/2$. But problem with anti-quenching factor $(1+t)^{-3}$

$$u_t=u_{xx}+\frac{1}{(1+t)^{-3}(1-u)},\quad u(0,x)=0$$

has the global solution

$$u = 1 - \sqrt{1 - \frac{(1+t)^2 - 1}{2(1+t)^2}}.$$

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