Interchanges and Invariant Positions in $\mathfrak{A}_C(R,S)$*

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Let $C=(c_{ij})$ be an $m\times n$ $(0,1)$-matrix. Let $R=(r_1,\cdots,r_m)$ and $S=(s_1,\cdots,s_n)$ be nonnegative integral vectors. Denote by $\mathfrak{A}_C(R,S)$ the set of all $m\times n$ $(0,1)$-matrices $A=(a_{ij})$ satisfying $a_{ij}\geqslant c_{ij}, a_{i1}+\cdots+a_{in}=r_i, a_{1j}+\cdots+a_{mj}=s_j$ for $1\leqslant i\leqslant m$, $1\leqslant j\leqslant n$.

Consider the following $t\times t$ matrices.

$$\text{i)}\ \begin{pmatrix} 0 & 1 & \text{①}'s \\ & \ddots & \ddots \\ \text{①}'s & 0 & 1 \end{pmatrix} \qquad \text{ii)}\ \begin{pmatrix} 1 & 0 & \text{①}'s \\ & \ddots & \ddots \\ \text{①}'s & 1 & 0 \end{pmatrix} \qquad (1)$$

where ① denotes element 1 of C. Replacement of a submatrix i) by ii) or vice versa leaves the row and column sums unchanged. A t-interchange is such aure-placement or any version of (1) obtained by applying the same row permutation to both i) and ii).

Theorem 1 Let u be the maximum of row and column sums of C. Given a pair $A,B\in\mathfrak{A}_C(R,S)$, one can get from A to B by a series of k-interchanges for $2\leqslant k\leqslant u+2$, without leaving $\mathfrak{A}_C(R,S)$.

The position (e,f) is an invariant 1 provided all of the matrices in $\mathfrak{A}_C(R,S)$ have their (e,f)—entry to 1, which is not 1 of C.

Theorem 2 Suppose (e,f) is an invariant 1 of $\mathfrak{A}_C(R,S)$. Then there exist $e\in I\subseteq\{1,\cdots,m\}$, $f\in J\subseteq\{1,\cdots,n\}$, such that for every matrix $A\in\mathfrak{A}_C(R,S)$, $A[I,J]$ is the matrix of I's, $A[I,J]\neq C[I,J]$ and $A[\bar{I},\bar{J}]=C[\bar{I},\bar{J}]$, where $\bar{I}=\{1,\cdots,m\}-I$ and $\bar{J}=\{1,\cdots,n\}-J$.⁺

Theorem 1 reduces to Ryser's Interchange Theorem[1] and Anstee's results[2] when $C=0$ and $C=P$, respectively. Theorem 2 generalizes a result of Ryser on $\mathfrak{A}(R,S)$[1].

References

[1] Ryser, H. J., Combinatorial Mathematics, Chapter 6, Carus Math. Monographs, No. 14, 1963.
[2] Anstee, R. P., Canad. J. Math., 34 (1982) 438—453.

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⁺$A[I,J]$ denotes the submatrix of A whose rows are indexed by I and whose columns are indexed by J.