第5卷 第1期
1985年1月

数学研究与评论
JOURNAL OF MATHEMATICAL RESEARCH AND EXPOSITION

Vol.5 No.1
Jan. 1985

Lacunary Interpolation by Spline (I)*

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Let $f(x)$ be defined on $[0,1]$, n be any positive integer and $[0,1]$ be subdivided into n equal subintervals. Set

$$h=\frac{1}{n},\ f^{(r)}(\nu h)=f_\nu^{(r)},\ \nu=0,1,\cdots,n,\ r=0,1,\cdots 5.$$

For odd n, A. Meir and A. Sharma [1] considered the lacunary spline interpolation as follows:

1) $S_n(x)\in C^3[0,1]$; 2) $S_n(x)\in\pi_5$, $x_\nu\in[\nu h,(\nu+1)h]$, $\nu=0,1,\cdots,n-1$;

3) $S_n(\nu h)=f_\nu$, $S_n''(\nu h)=f_\nu''$, $\nu=0,1,\cdots,n$; 4) $S'''(0)=f_0'''$, $S_n'''(1)=f_n'''$.

[1] proved that $S_n(x)$ was uniquely determined, and gave the error bounds of approximation of $f(x)\in C^4[0,1]$ by $S_n(x)$. B. K. Swartz and R. S. Varga [2] also gave the degree of approximation of $f(x)\in C^6[0,1]$ by $S_n(x)$. Z. R. Guo [3] obtained the saturation theorem for this approximation. Z. R. Guo [4-5] were also concerned in this kind of interpolation and gave the degree of approximation. In this paper we consider the general lacunary interpolation by quintic splines.

Let Δ: $0=x_0<x_1<\cdots<x_n=1$ be a partition of the interval $[0,1]$ and let S be the set of all functions $S(x)$ satisfying

(i) $S(x)\in C^3[0,1]$; (ii) $S(x)\in\pi_5$, $x\in[x_i,x_{i+1}]$, $i=0,1,\cdots,n-1$.

Put $T=\{0,1,2,3\}$, z_{1i}, $z_{2i}\in T$, $z_{1i}<z_{2i}$. For any given $f(x)$, if $S(x)\in S_\Delta$, $S^{(z_{1i})}(x_i)=f^{(z_{1i})}(x_i)$ and $S^{(z_{2i})}(x_i)=f^{(z_{2i})}(x_i)$, $i=0,1,\cdots,n$, then we denote this kind of interpolation conditions by

$$Z=\begin{pmatrix} z_{20} & z_{21}\cdots z_{2n} \\ z_{10} & z_{11}\cdots z_{1n}\end{pmatrix}.$$

Furthermore, we consider two additional interpolation conditions. Let

$$z'\in T\backslash\{z_{1i},z_{2i}\},\ z''\in T\backslash\{z_{1j},z_{2j}\},\ S^{(z')}(x_i)=f^{(z')}(x_i),\ S^{(z'')}(x_j)=f^{(z'')}(x_j),$$

and denote these condition by $b(x_i,z';x_j,z'')$.

We say an interpolation to be of type I, if the corresponding interpolation conditions are

* Received Jan. 3, 1983.

$$\tilde{Z}=Z+b(x_0,z';x_n,z'');$$

to be of type II, if

$$\tilde{Z}=Z+b(x_i,z';x_j,z''),$$

where $0\leqslant i<j\leqslant n$ and the equalities do not hold simultaneously here; to be of type III, if

$$\tilde{Z}=Z+b(x_i,z';x_i,z''),\quad \{z_{1i},z_{2i},z',z''\}=T.$$

we denote each type of these interpolations by $S(x)|_{\tilde{Z}}$.

At first, we consider the case of equidistant knots

$$x_i=\frac{i}{n}=ih,\quad i=0,1,\cdots,n.$$

Set $z_i\in T$, $i=1,2,3,4$ and $z_2>z_1$, $z_4>z_3$; $\{i_1,i_2\}=T\backslash\{z_1,z_2\}$, $i_1<i_2$;

$$\{j_1,j_2\}=T\backslash\{z_3,z_4\},\quad j_1<j_2.$$

For any given α and β let $p(x)\in\pi_5$ be the polynomial uniquely determined by the conditions

$$p^{(z_1)}(0)=p^{(z_2)}(0)=p^{(z_3)}(1)=p^{(z_4)}(1)=0,\quad p^{(i_1)}(0)=\alpha,\quad p^{(i_2)}(0)=\beta.$$

Set

$$\alpha'=p^{(j_1)}(1),\quad \beta'=p^{(j_2)}(1).$$

The matrix $B_{\binom{z_2z_4}{z_1z_3}}$ stisfying $(\alpha',\beta')=(\alpha,\beta)B_{\binom{z_2z_4}{z_1z_3}}$ is called a T-matrix.

We have 36 T-matrices, they are all nonsingular.

Let

$$(k,l)=(\varphi(z'),\psi(z'')),$$

where

$$\varphi(i)=\begin{cases}1, & i=\max(T\backslash\{z_{20},z_{10}\})\\ 2, & i=\min(T\backslash\{z_{20},z_{10}\})\end{cases};\quad \psi(i)=\begin{cases}1, & i=\min(T\backslash\{z_{1n},z_{2n}\})\\ 2, & i=\max(T\backslash\{z_{1n},z_{2n}\}).\end{cases}$$

We prove

Theorem 1 Let

$$B_2=B_{\binom{z_{20}z_{21}}{z_{10}z_{11}}}B_{\binom{z_{21}z_{22}}{z_{11}z_{12}}}\cdots B_{\binom{z_{2n-1}z_{2n}}{z_{1n-1}z_{1n}}}=(b_{ij}).\tag{1}$$

A necessary and sufficient condition for the interpolation conditions of type I

$$\tilde{Z}=Z+b(x_0,z';x_n,z'')\tag{2}$$

to be regular is $b_{kl}\neq0$.

proof We denote zero matrix of two rows by 0. Set

$$\{m_{1i},m_{2i}\}=T\backslash\{z_{1i},z_{2i}\},\quad m_{1i}<m_{2i},\quad i=0,1,\cdots,n.$$

In the interval $[x_0,x_1]$ let $t=\dfrac{x-x_0}{h}$, then for any given α_0 and β_0 there exists an unique $p_0(t)\in\pi_5$ such that

$$p_0^{(z_{10})}(0)=p_0^{(z_{20})}(0)=p_0^{(z_{11})}(1)=p_0^{(z_{21})}(1)=0,\quad p_0^{(m_{10})}(0)=\alpha_0,\quad p_0^{(m_{20})}(1)=\beta_0.$$

By definition of T-matrix, we have

$$(\alpha_1,\beta_1)=(\alpha_0,\beta_0)B_{\binom{z_{20}z_{21}}{z_{10}z_{11}}},$$

where $\alpha_1=p_0^{(m_{11})}(1)$, $\beta_1=p_0^{(m_{21})}(1)$.

In general, we have

$$(\alpha_i,\beta_i) = (\alpha_0,\beta_0)B_{Z(i)}, \quad i = 1,2,\cdots,n \tag{3}$$

where

$$B_{Z(i)} = B_{\binom{z_{10}z_{21}}{z_{10}z_{11}}} B_{\binom{z_{11}z_{21}}{z_{11}z_{21}}} \cdots B_{\binom{z_{1i-1}z_{2i}}{z_{1i-1}z_{1i}}}. \tag{4}$$

Put $i=n$ in (3), (4) and combine (1) we obtain

$$\alpha_n = b_{11}\alpha_0 + b_{21}\beta_0, \quad \beta_n = b_{12}\alpha_0 + b_{22}\beta_0. \tag{5}$$

Thus (α_0,β_0) uniquely determines those splines which satisfy the conditions $S(x)|_Z = 0$.

Now consider the boundary interpolation conditions $b(x_0,z';x_n,z'')$. Suppose $z' = m_{10}$ and $z'' = m_{1n}$. Thus, $\alpha_0 = \alpha_n = 0$ determine the spline which satisfy the conditions

$$S(x)|_{Z+b(x_0,m_{10};x_n,m_{1n})} = 0. \tag{6}$$

If $b_{21}\neq 0$ from (5) we obtain $\beta_0 = \beta_n = 0$, consequently $S(x)\equiv 0$. Thus the interpolation condition (2) are regular;

In the other hand, if $b_{21} = 0$, then we may put $\alpha_0 = 0$ and arbitrary $\beta_0 \neq 0$ to determine $p(t)$, thus we obtain a nonzero spline which satisfies (6). Therefore, the interpolation conditions (6) are not regular.

We can prove the case of the other boundary interpolation conditions similarly. Theorem I established;

Since the matrix B_Z is the product of nonsingular matries $B_{\binom{z_{1i}\ z_{2(i+1)}}{z_{1i}\ z_{1(i+1)}}}$, so itself is nonsingular. Thus we obtain

Corollary If the interpolation conditions

$$\tilde{Z} = Z + b(x_0,z_0';x_n,z_n')$$

are nonregular, let

$$z_0'' = T\backslash\{z_{10},z_{20},z_0'\}, \quad z_n'' = T\backslash\{z_{1n},z_{2n},z_n'\},$$

then the interpolation conditions

$$Z + b(x_0,z_0'';x_n,z_n') \text{ and } Z + b(x_0,z_0';x_n,z_n'')$$

must be regular;

This means that if the interpolation of type I is nonregular, then, by changing any one of the boundary conditions, the regular interpolation can be obtained.

Theorem 2 A necessary and sufficient condition for the interpolation conditions of type II

$$\tilde{Z} = Z + b(x_i,z';x_j,z''), \quad 0\leqslant i<j\leqslant n$$

to be regular is the interpolation conditions of type I

$$\mathbf{Z} = \binom{z_{2i}\cdots z_{2j}}{z_{1i}\cdots z_{1j}} + b(x_i,z';x_j,z'')$$

to be regular.

Proof Suppose the interpolation conditions are regular, then the spline $S(x)$ such that $S(x)|_{\tilde{Z}}=0$ must vanish on $[x_i,x_j]$, thus $\alpha_j=\beta_j=0$, therefore $S(x)$ vanishes on $[x_j,x_n]$. Similarly, $\alpha_i=\beta_i=0$ deduce $S(x)$ also vanishes on $[x_0,x_i]$.

On the other hand, if the interpolation conditions are nonregular, then there exists nonzero spline $S(x)$ such that $S(x)|_{\tilde{Z}}=0$, thus, α_j, β_j does not equal to zero simultaneously. As the proof of theorem I, we can extend this spline $S(x)$ to $[x_j,x_n]$ as well as to $[x_0,x_i]$ such that $S(x)|_{\tilde{Z}}=0$. Theorem 2 established.

Obviously, we have

Theorem 3 The interpolation conditions of type III

$$\tilde{Z}=Z+b(x_i,z';x_i,z'')$$

are always regular.

As for the case of unequally distributed knots, a T-matrix depends on the length h of the subinterval, we denote it by $B_{\binom{z_1 z_4}{z_2 z_3}}(h)$.

On the interval $[0,h]$, the conditions

$$p^{(z_1)}(0)=p^{(z_2)}(0)=p^{(z_3)}(h)=p^{(z_4)}(h)=0;\quad p^{(i_1)}(0)=\alpha,\quad p^{(i_2)}(0)=\beta$$

uniquely determine $p(t)\in\pi_5$. Let

$$\alpha'=p^{(j_1)}(h),\quad \beta'=p^{(j_2)}(h)\ \text{and}\ (\alpha',\beta')=(\alpha,\beta)\ B_{\binom{z_1 z_4}{z_2 z_3}}(h).$$

Evidently

$$B_{\binom{z_1 z_4}{z_2 z_3}}(h)=\begin{pmatrix} b_{11}h^{i_1-j_1} & b_{12}h^{i_1-j_2} \\ b_{21}h^{i_2-j_1} & b_{22}h^{i_2-j_2}\end{pmatrix},\quad \text{if}\quad B_{\binom{z_1 z_4}{z_2 z_3}}=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}.$$

In this case, theorem 1 and its corollary still hold, whereas

$$B_Z=B_{\binom{z_{10}z_{11}}{z_{10}z_{11}}}(h_1)B_{\binom{z_{21}z_{11}}{z_{11}z_{11}}}(h_2)\cdots B_{\binom{z_{1n-1}\ z_{1n}}{z_{1n-1}\ z_{1n}}}(h_n)=(b_{ij}).$$

Similarly, Theorems 2,3 hold for the case of unequally distributed knots.

Reference

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