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Admissible Estimation of Parameters in Linear Model*

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Abstract

This paper considers the estimation of $f'\theta$, $\beta' c\beta$ and their linear combination by quadratic estimates in variance components model. The solution to the LaMotte's Problem (see[1]) and some interesting results on the admissible or inadmissible estimation of quantities involving θ and β are presented.

§1 Introduction

Consider variance components model

$$\overset{n\times 1}{Y} = \overset{n\times k}{X}\overset{k\times 1}{\beta} + \overset{n\times 1}{\varepsilon}, \quad E\varepsilon\varepsilon' = \theta_1 V_1 + \cdots + \theta_p V_p = V_\theta \tag{1.1}$$

where $\beta\in R^k$, $\theta\in\Theta$, $\Theta\cap\{\theta;\ V_\theta>0\}\neq\emptyset$. Let the risk function of $Y'AY$ be $R(A, \theta, \beta) = E(Y'AY - f'\theta - \beta' c\beta)^2$. $Y'AY$ is said to be better than $Y'BY$ iff $R(A,\theta,\beta) \leqslant R(B,\theta,\beta)$ for all $\beta\in R^k$ and $\theta\in\Theta$ and the inequality holds at least for one pair of $(\beta_0, \theta_0)\in R^k\times\Theta$; $Y'AY$ is said to be admissible among $\mathscr{B}$ which is a subset of quadratic estimates iff there is no $Y'BY\in\mathscr{B}$ such that $Y'BY$ is better than $Y'AY$.

§2 Inadmissible Quadratic Estimate of Variance Components

Suppose that the variance components model is (1.1) and Y is normally distributed. Under these assumptions, it has been verified (see[1]) that if $Y'AY$ is invariant unbiased for $f'\theta$, $Y'AY$ is inadmissible among the class of invariant quadratic estimates for $f'\theta$. But he didn't establish whether $Y'AY$ is inadmissible among the class of quadratic estimates or not while Y is normally distributed.

Theorem 2.1 Suppose that the model considered here is (1.1) and Y is elliptically contoured distributed, i.e., for each $\beta\in R^k$ and $\theta\in\Theta$, there exist R and $u^{(r)}$ such that $Y\overset{d}{=}X\beta + RB'u^{(r)}$, where $R\geqslant 0$ and $u^{(k)}$ are independent, $u^{(r)}$ is uniformly

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distributed on the unit sphere in R^r, $0<ER^4<\infty$, $r=\mathrm{rank}(V_\theta)$, $V_\theta=\frac{ER^2}{r}B'B$; R, r and B can depend on the parameters; the notation "$X\overset{d}{=}Y$" means that X and Y have the same distributions. Assume further that $d=\frac{rER^4}{(r+2)(ER^2)^2}\geqslant 1$ and $Y'AY$ is unbiased for $f'\theta$. Then $Y'AY$ is inadmissible among the class of quadratics. Furthermore, there exists a class of estimates of the form $cY'AY(c\in R')$ such that $cY'AY$ is better than $Y'AY$.

Proof $R(cA,\ \theta,\ \beta)-R(A,\theta,\beta)=2(c^2-1)d\mathrm{tr}AV_\theta AV_\theta+(c^2-1)(d-1)(\mathrm{tr}AV_\theta)^2$
$+(c-1)^2(\mathrm{tr}AV_\theta)^2+4(c^2-1)\beta'X'AV_\theta AX\beta\leqslant 2(c^2-1)\mathrm{tr}AV_\theta AV_\theta+n(c-1)^2\mathrm{tr}AV_\theta AV_\theta$
$\leqslant 0,\ \forall\ \beta\in R^k,\ \theta\in\Theta$

where $d\geqslant 1$ and $c\in\left\{c: |c|<1 \text{ and } c>\frac{n-2}{n+2}\right\}$. This implies that $cY'AY$ is better than $Y'AY$ and hence $Y'AY$ is inadmissible.

Note That Y is normally distributed implies $d=1$. So theorem 2.1 holds in normal case. It is easy to see that the above proof also fits for the invariant case, so the result of LaMotte is simply a corollary of this theorem.

§3 Estimation of Quantities Involving θ and β

Let's still consider model (1.1), and the quadratic estimation of $\gamma=f'\theta+\beta'c\beta$. It is easy to see that $Y'AY$ is unbiased for γ iff $C=X'AX$ and $\mathrm{tr}AV_i=f_i$, $i=1,\cdots,p$.

Theorem 3.1 Let Y be normally distributed. Then γ is quadratic unbiased estimable iff γ is unbiased estimable.

Proof Let $T(Y)$ be unbiased for γ and $\theta_0\in\Theta\cap\{\theta: V_\theta>0\}$. Choose $a>0$ and $b>0$ such that $a+b=1$ and $\frac{1}{a}\theta_0\in\Theta\cap\{\theta: V_\theta>0\}$. Since $\|Y-X\beta\|^m\exp\left\{-\frac{b}{2}(Y-X\beta)'\cdot V_{\theta_0}^{-1}(Y-X\beta)\right\}$ is bounded abovs, where $m=1,2,\cdots$. Thus we have

$$\frac{\partial}{\partial\beta}\left(\frac{\partial}{\partial\beta}\right)'E(T(Y))=\frac{\partial}{\partial\beta}\left(\frac{\partial}{\partial\beta}\right)'(f'\theta+\beta'c\beta)=2c=X'\left\{\text{const.}\int T(Y)[V_{\theta_0}^{-1}(Y-X\beta)(Y-X\beta)'V_{\theta_0}^{-1}-V_{\theta_0}^{-1}]\exp\left\{-\frac{1}{2}(Y-X\beta)'V_{\theta_0}^{-1}(Y-X\beta)\right\}dY\right\}X.$$

So there must be some C_0 such that $C=X'C_0X$. By Pincus Theorem (see [2])*), $f'\theta-\mathrm{tr}PC_0PV_\theta$ is quadratic unbiased estimable, so is γ, here $P=XX^+$.

Lemma 3.2 Let $Y\sim N_n(X\beta,\sigma^2I)$, C be a symmetric matrix such that $\mathscr{U}(C)\subset$

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*) Mr. Tian-shi Su has given a new proof of this theorem by means of differentiation with respect to θ.

$\mathscr{U}(X')$. Then the MVUE of $b\sigma^2+\beta'c\beta$ is $\hat\beta'c\hat\beta-[b-\mathrm{tr}C(X'X)^-]\hat\sigma^2$, where $\hat\beta=(X'X)^-X'Y$, $\hat\sigma^2=\frac{1}{n-r}Y'MY$, $r=\mathrm{rank}(X)$, $M=I-P$, $\mathscr{U}(X)$ denotes the column space of X.

Proof The density function of Y can be expressed as

$$(2\pi)^{-n/2}\sigma^{-n}\exp\left\{-\frac{1}{2\sigma^2}\left(\|Y\|^2+\|X\beta\|^2-2\beta'X'Y\right)\right\}$$

which belongs to the exponential family. Hence, $(\|Y\|^2, X'Y)$ is complete and sufficient. That $\hat\beta'c\hat\beta+[b-\mathrm{tr}C(X'X)^-]\hat\sigma^2$ is unbiased for $b\sigma^2+\beta'c\beta$ and the Lehmann-Scheffe Theorem imply that $\hat\beta'c\hat\beta+[b-\mathrm{tr}C(X'X)^-]\hat\sigma^2$ is the MVUE of $b\sigma^2+\beta'c\beta$.

Theorem 3.3 Suppose that the model is (1.1) and Y is normally distributed, $\theta_0\in\Theta\cap\{\theta: V_\theta>0\}$, and $\gamma=f'\theta+\beta'c\beta$ is unbiased estimable. Then the LMVUE at (β_0, θ_0) of γ is $\hat\gamma=YV_{\theta_0}^{-1}X(X'V_{\theta_0}^{-1}X)^-C(X'V_{\theta_0}^{-1}X)^-X'V_{\theta_0}^{-1}Y+(Y-X\beta_0)'\sum_1^p\lambda_iV_{\theta_0}^{-1}(V_i-P_{\theta_0}V_iP'_{\theta_0})V_{\theta_0}^{-1}(Y-X\beta_0)$, where $P_{\theta_0}=X(X'V_{\theta_0}^{-1}X)^-X'V_{\theta_0}^{-1}$, $\lambda=(\lambda_1,\cdots,\lambda_p)'$ satisfies $(\mathrm{tr}V_{\theta_0}^{-1}\cdot(V_i-P_{\theta_0}V_iP'_{\theta_0})V_{\theta_0}^{-1}V_j)\lambda=f-g$, $g_i=\mathrm{tr}V_{\theta_0}^{-1}X(X'V_{\theta_0}^{-1}X)^-C(X'V_{\theta_0}^{-1}X)^-X'V_{\theta_0}^{-1}V_i$, $i=1,\cdots,p$.

Note This theorem can be verified by applying Lemma 3.2, Theorem 4.1.1 in [2] and the general theorem in estimation theory.

We have see in section 2 that if one only considers the estimation of variance components, the unbiased quadratic estimator is inadmissible in the above sense. Whether is it true for the estimation of quantities involving θ and β? In the following, we will present some necessary and sufficient or sufficient conditions for this. It is well known that an estimate $T(Y)$ is not always the admissible estimate for its expectation, and in some cases, its shrunken estimate is better than $T(Y)$. This method is very powerful in some problems such as Stein's and LaMotte's. But there are some exceptions as follows.

Lemma 3.4 Let the model be (1.1) and Y be normally distributed. Suppose A is a symmetric matrix such that $X'AX\neq0$. Then there is no shrunken estimate of $Y'AY$ such that it is better than $Y'AY$.

Proof Notice that $R(cA,\theta,\beta)-R(A,\theta,\beta)=(c^2-1)(2\mathrm{tr}AV_\theta AV_\theta+4\beta'X'AV_\theta AX\beta)+(c-1)^2(\mathrm{tr}AV_\theta+\beta'X'AX\beta)^2\leqslant0$ can not hold for all $\beta\in R^k$ and a fixed $\theta\in\Theta$ while $X'AX\neq0$.

Theorem 3.5 Suppose that $Y\sim N_n(X\beta,\sigma^2I)$——the variance components model with $p=1$ and $\theta_1=\sigma^2$ and A is a symmetric matrix such that $X'AX\neq0$. Then we have

(i) $\mathscr{U}(A)\not\subset\mathscr{U}(X)$ implies that $Y'AY$ is inadmissible among the quadratic estimates for its expectation;

(ii) $\mathscr{U}(A)\subset\mathscr{U}(X)$ implies that $Y'AY$ is admissible among the quadratic estimates for its expectation.

Proof (i) $\mathscr{U}(A)\not\subset\mathscr{U}(X)$ implies that $A-PAP\neq 0$, where $P=XX^{+}$. Let $B_\delta=A+\delta(A-PAP)$, $\delta\in R^1$. Then

$$R(B_\delta,\sigma^2,\beta)-R(A,\sigma^2,\beta)=\sigma^4\{4\delta\mathrm{tr}(A-PAP)^2+2\delta^2\mathrm{tr}(A-PAP)^2+\delta^2[\mathrm{tr}(A-PAP)]^2\}$$
$$+4\delta\sigma^2(2+\delta)\mu'(A-PAP)^2\mu<0,\ \forall\beta\in R^k,\ \sigma^2>0.$$

where $\delta\in\left(\max\left[-2,\ -\frac{4\mathrm{tr}(A-PAP)^2}{2\mathrm{tr}(A-PAP)^2+[\mathrm{tr}(A-PAP)]^2}\right],\ 0\right)$, $\mu=X\beta$. This implies that $Y'B\delta Y$ is better than $Y'AY$ and $Y'AY$ is inadmissible.

(ii) Suppore that $\mathscr{U}(A)\subset\mathscr{U}(X)$ but $Y'AY$ is inadmissible among the quadratic estimates, i.e., there is a symmetric matrix D such that

$$R(A+D,\sigma^2,\beta)-R(A,\sigma^2,\beta)=4\sigma^4\mathrm{tr}DA+8\sigma^2\mu'DA\mu+2\sigma^4\mathrm{tr}D^2+4\sigma^2\mu'D^2\mu+(\sigma^2\mathrm{tr}D+\mu'D\mu)^2\leqslant 0,\ \forall\beta\in R^k,\ \sigma^2\geqslant 0,\ \mu=X\beta, \tag{3.1}$$

and the inequality holds at least for some $(\beta_0,\sigma_0^2)\in R^k\times\Theta$. This implies that $PDP=0$, since the term with the highest degree of μ appears in the last term in (3.1). By using $PDP=0$, we get another expression

$$R(A+D,\sigma^2,\beta)-R(A,\sigma^2,\beta)=4\sigma^4\mathrm{tr}D(PAP)+2\sigma^4\mathrm{tr}D^2+\sigma^4(\mathrm{tr}D)^2+8\sigma^2\alpha'PD(PAP)\mu$$
$$+4\sigma^2\mu'D^2\mu=2\sigma^4\mathrm{tr}D^2+\sigma^4(\mathrm{tr}D)^2+4\sigma^2\mu'D^2\mu\geqslant 0,\qquad\forall\beta\in R^k,\ \sigma^2\geqslant 0, \tag{3.2}$$

where $\mu=X\beta=P\alpha$. This contradicts the assumption that $Y'(A+D)Y$ is better than $Y'AY$. So $Y'AY$ is admissible among the quadratic estimates for its expectation.

Note $\mathscr{U}(A)\subset\mathscr{U}(X)$ implies that $A=PAP$, where A is symmetric and $P=XX^{+}$.

Theorem 3.6 Let $Y\sim N_n(\beta 1_n,\theta_1 1_n1_n'+\theta_2 I)$ —— the simplest Balanced One-Way classification ANOVA model, where $\theta_1\geqslant 0$, $\theta_2>0$. Suppose further that $Y'AY$ is unbiased for $\gamma=f'\theta+b\beta^2$, here $b\neq 0$. Then we have

(i) $\mathscr{U}(A)\not\subset\mathscr{U}(1_n)$ implies that $Y'AY$ is inadmissible among the quadratic estimates of γ;

(ii) $\mathscr{U}(A)\subset\mathscr{U}(1_n)$ implies that $Y'AY$ is admissible among the quadratic estimates of γ.

Proof After calculating one can get $P=\frac{1}{n}1_n1_n'$ and $PV_\theta=(n\theta_1+\theta_2)P$.

(i) $\mathscr{U}(A)\not\subset\mathscr{U}(1_n)$ implies that $A-PAP\neq 0$. Let $B_\delta=A+\delta(A-PAP)$, $\delta\in R'$. Then

$$R(B_\delta,\theta,\beta)-R(A,\theta,\beta)=2\delta\{2\mathrm{tr}AV_\theta(A-PAP)V_\theta+4\beta^2 1_n'AV_\theta(A-PAP)1_n\}+$$
$$\delta^2\{2\mathrm{tr}(A-PAP)V_\theta(A-PAP)V_\theta+4\beta^2 1_n'(A-PAP)V_\theta(A-PAP)1_n+[\mathrm{tr}(A-PAP)V_\theta]^2\}\leqslant$$
$$(4\delta+2\delta^2+n\delta^2)\mathrm{tr}(A-PAP)V_\theta(A-PAP)V_\theta+4\beta^2(2\delta+\delta^2)1_n'(A-PAP)V_\theta(A-PAP)1_n$$
$$<0,\ \forall\beta\in R^1,\ \theta_1\geqslant 0,\ \theta_2>0,$$

where $\delta\in\left(\max\left[-2,\frac{-4}{2+n}\right],0\right)$. This implies that $YA'Y$ is inadmissible. For the proof of (ii), one can see the following theorem.

Theorem 3.7 Suppose that the model is (1.1) and Y is normally distributed, $\alpha \in \Theta \cap \{\theta: V_\theta > 0\}$, A is a symmetric matrix such that $X'AX \neq 0$ and $\mathscr{U}(V_\alpha^{1/2}AV_\alpha^{1/2}) \subset \mathscr{U}(V_\alpha^{-1/2}X)$. Then $Y'AY$ is admissible among the quadratic estimates for its expectation.

Proof The inadmissibility of $Y'AY$ implies that there exists a symmetric matrix D such that

$$R(A+D,\alpha,\beta) - R(A,\alpha,\beta) = 4\mathrm{tr}AV_\alpha DV_\alpha + 8\beta'X'AV_\alpha DX\beta + 2\mathrm{tr}DV_\alpha DV_\alpha + 4\beta'X'DV_\alpha DX\beta + (\mathrm{tr}DV_\alpha + \beta'X'DX\beta)^2 \leqslant 0, \quad \forall \beta \in R^k. \tag{3.3}$$

This implies that $X'DX=0$ or $PDP=0$. So we have another expression

$$R(A+D,\alpha,\beta) - R(A,\alpha,\beta) = 2\mathrm{tr}(V_\alpha^{1/2}DV_\alpha^{1/2})^2 + (\mathrm{tr}DV_\alpha)^2 + 4\beta'X'V_\alpha^{-1/2}(V_\alpha^{1/2}DV_\alpha^{1/2})^2V_\alpha^{-1/2}X\beta > 0, \quad \forall \beta \in R^k \tag{3.4}$$

This contradicts the assumption that $Y'AY$ is inadmissible. Thus, $Y'AY$ is admissible among the quadratic estimates for its expectation.

Corollary 3.8 Let $Y \sim N_n(X\beta, \sigma^2 I)$, and C be a symmetric matrix such that $\mathscr{U}(C) \subset \mathscr{U}(X')$, and $b \in R^1$ such that $b\text{-}\mathrm{tr}C(X'X)^- \neq 0$. Under these assummptions, the MVUE of $b\sigma^2 + \beta'c\beta$ is inadmissible among the quadratic estimates.

Proof By Lemma 3.2, the MVUE of $b\sigma^2 + \beta'c\beta$ is

$$Y'\left\{X(X'X)^-C(X'X)^-X' + \frac{b-\mathrm{tr}C(X'X)^-}{n-r}M\right\}Y$$

which meets the condition in (ii) of Theorem 3.5.

Corollary 3.9 Let $Y \sim N(X\beta, \sigma^2 I)$, $\mathscr{U}(D) \subset \mathscr{U}(X')$, $\gamma = b\sigma^2 + \beta'D\beta$. Then (a) $b\text{-}\mathrm{tr}D(X'X)^- \neq 0 \Longrightarrow$ every unbiased quadratic estimator of γ is inadmissible; (b) $b = \mathrm{tr}D(X'X)^- \Longrightarrow$ there is one and only one admissible quadratic estimator which is also unbiased for γ and this estimator is $\hat{\beta}'D\beta$.

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References

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