A Combinatorial Problem*

Proposed by
K. L. Chung
(Stanford, U. S. A.)

Let a and b be two positive integers, $a<b$. Let $P_0=1$ and for $n\geqslant 1$:

$$P_n=\binom{nb}{na}\left(\frac{a}{b}\right)^{n(b-a)}\left(\frac{b-a}{b}\right)^{na}.$$

Define $\{f_n, n\geqslant 1\}$ by the recursive formula $P_n=\sum_{k=1}^n f_k P_{n-k}$. Find the explicit formula for f_n. For $a=1$, the answer is given by

$$f_n=(bn-2)!\,b^{1-bn}\cdot(b-1)^{n+1}/(bn-n)!\,(n-1)!.$$

Note that f_n has the following meaning: Let X_n, $n\geqslant 1$ be a Bernoullian random walk with $P(X_n=b-a)=a/b$, $P(X_n=-a)=(b-a)/b$. If $S_n=\sum_{k=1}^n X_k$, then $f_n=P(S_k\neq 0,\ 1\leqslant k<bn;\ S_{bn}=0)$.

Note added by L. C. Hsu Starting with the generating function

$$\sum_{n=1}^{\infty} f_n t^n=\left(\sum_{k=1}^{\infty} P_k t^k\right)\Big/\left(1+\sum_{j=1}^{\infty} P_j t^j\right),$$

one may determine f_n by means of Bell's polynomial expression. More precisely we have

$$f_n=d\sum_{(J)}(-1)^{k-1}\frac{k!}{j_1!\,j_2!\cdots j_n!}\binom{1b}{1a}^{j_1}\binom{2b}{2a}^{j_2}\cdots\binom{nb}{na}^{j_n},$$

where $d=(a/b)^{n(b-a)}((b-a)/b)^{na}$ and (J) denotes the summation condition (J): $j_1+j_2+\cdots+j_n=k$, $1j_1+2j_2+\cdots+nj_n=n$, $j_k\geqslant 0$, $k=1,2,\cdots,n$. Let δ_n denote the "incomplete Kronecker delta symbol" such that $\delta_n^m=\delta_n\cdot\delta_n\cdots\delta_n(m \text{ times})=0$ or $m\neq n$, and $\delta_n^m=1$ for $m=n$. Then the summation may be put in the more condensed form

$$f_n=d\sum_{k=1}^{n}(-1)^{k-1}\left(\sum_{r=1}^{n}\binom{rb}{ra}x^r\right)^k_{x=\delta_n}.$$

This formula is actually impractical for large n. However it seems not easy to get a closed form except for the particular case $a=1$. We hope the interested reader would like to try his hand at the evaluation or simplification of the summation mentioned above.

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