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Uniqueness of Solutions to Conservation Laws with Linearly Degenerate Eigenvalue*

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Consider a hyperbolic system of conservation laws

$$u_t + f(u)_x = 0, \tag{1}$$

where $u = (x, y) \in R^2, f: R^2 \to R^2$ is a smooth nonlinear mapping. ∇f has two real and distinct eigenvalues λ_1 and $\lambda_2 (\lambda_1 < \lambda_2)$, one of which is genuinely nonlinear, i.e.

$$\nabla\lambda_j(u) r_j(u) \neq 0, \quad j = 1 \text{ or } 2, \tag{2}$$

the other one of which is linearly degenerate in sense of Lax[2], i, e,

$$\nabla\lambda_k(u) r_k(u) \equiv 0, \quad k \neq j, \tag{3}$$

It is well known that in general the initial value problem for (1) does not have globally defined smooth solutions. On the other hand, uniqueness is lost in the broader class of weak solutions. An important work on uniqueness of solutions to conservation laws (1) was done by R. J. Diperna [1]. He proved that each piecewise Lipschitz continuous solution of system (1) is unique within admissible weak solutions (i. e. bounded BV solutions satisfying the entropy condition) when both two eigenvalues are genuinely nonlinear. But for the system (1) with eigenvalues of (2) and (3), he obtained only that the classical solution to the Riemann problem is unique within admissible weak solutions. In this paper we improve the latter result and prove that each piecewise Lipschitz continuous solution to the general initial value problem of system (1) with eigenvalues of (2) and (3) is unique within solutions of that class as well.

Definition 1 A function $\eta: \mathscr{D} \to R$ defined on an open domain $\mathscr{D} \subset R^2$ is called an entropy with entropy flux $q: \mathscr{D} \to R$, if all smooth solutions with range in $\mathscr{D}$ satisfy

$$\eta(u)_t + g(u)_x = 0. \tag{4}$$

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Definition 2 Let $\Omega\subset R^2$ be an open domain. A function $u\in L^\infty(\Omega)\cap BV(\Omega)$: $\Omega\to\mathscr{D}$ is called an admissible weak solution if it satisfies both (1) and

$$\eta(u)_t+q(u)_x\leqslant 0, \tag{5}$$

in the sense of distributions, where η is a strictly convex entropy.

We define

$$\theta_u\equiv\eta(u)_t+q(u)_x,$$
$$\gamma(u,v)\equiv\alpha(u,v)_t+\beta(u,v)_x,$$

where u and v are arbitrary two admissible weak solutions of (1) and

$$\alpha(u,v)\equiv\eta(u)-\eta(v)-\nabla\eta(v)(u-v):\ \mathscr{D}\times\mathscr{D}\to R,$$

$$\beta(u,v)\equiv q(u)-q(v)-\nabla\eta(v)(f(u)-f(v)):\ \mathscr{D}\times\mathscr{D}\to R.$$

It is easy to see that either θ_u or $\gamma(u,v)$ is a Borel measure defined on Ω. Denote

$$d(\tau,u_l,u_r)\equiv\tau[\eta]-[q]:R\times R^4\to R,$$

$$D(\tau,u_l,u_r,v_l,v_r)\equiv\tau[\alpha]-[\beta]:R\times R^4\times R^4\to R,$$

where $u_l(v_l)$ and $u_r(v_r)$ represent the approximate left and right hand limits of $u(v)$ at a point, and $[\eta]=\eta(u_l)-\eta(u_r)$, etc.

Let $K=K\{\varphi(T)\}$ and $PL=PL\{\varphi(T)\}$ denote the class of admissible weak solutions and the class of admissible piecewise Lipschitz continuous solutions respectively defined on the strip $\varphi(T)=\{(x,t):0\leqslant t<T\}$. We note that PL forms a broad subclass of K.

The main result of this paper is

Theorem Consider system (1) with eigenvalues of (2) and (3). For every $\tilde{u}\in R^2$ there exists a constant $\delta>0$ depending only on $\tilde{u}$ and f with the following property: If $u\in K\{\varphi(T)\}, w\in PL\{\varphi(T)\}$, $|u-\tilde{u}|_\infty<\delta$, $|w-\tilde{u}|<\delta$ and $u(x,0)=w(x,0)$ for almost all x, then $u=w$ for allmost all (x,t) in $\varphi(T)$.

Proof For concreteness we suppose λ_1 is linearly degenerate, i. e. $\nabla\lambda_1 r_1=0$. As in [1], for simplicity let us assume there is only a 1-contact discontinuity S_w^1: $x=y(t)$ in w, $\frac{dy(t)}{dt}=\tau(t)$. Thus we have

$$\tau(w_l-w_r)=f(w_l)-f(w_r),$$
$$\tau=\lambda_1(w_l)=\lambda_1(w_r). \tag{6}$$

Let $S_u^1: x=x(t)$ denote the minimal forward 1-charateristic in u. It follows from Theorem 3.2 of [1] that u and w coincide on

$$\{(x,t):x<\min[x(t),y(t)]\}.$$

Denote

$$E_0=\{t: x(t)=y(t), t\in[0,T)\},$$

$$E_1=\{t: x(t)<y(t), t\in[0,T)\},$$

$$E_2=\{t: x(t)>y(t), t\in[0,T)\}.$$

Obviously, sets E_0, E_1 and E_2 disjoint each other, and $[0,T)=E_0\cup E_1\cup E_2$; either E_1 or E_2 is open set, and can be regarded as an at most countable union of disjoint open intervals I_i or J_i:

$$E_1=\bigcup_{i=1}^{\infty} I_i,\quad E_2=\bigcup_{i=1}^{\infty} J_i.$$

By a property of BV functions[3], we know that there exist normals almost everywhere along $S_u^1\cap S_w^1$. Let $\tilde{E}_0$ denote the set of all such points at which the normals exist. Then $E_0\setminus\tilde{E}_0$ is a set with measure zero.

Similar to [1], we need only to research the structure of measure $\gamma(u,w)$ on $\varphi(T)$.

1° $t\in I_i\subset E_1, x(t)<y(t)$. Since λ_1 is linearly degenerate, we have

$$\lambda_1(u_l(P))=\frac{dx(t)}{dt}=\lambda_1(w_l(P))=\lambda_1(w_1(P)),$$

where $P\in S_u^1\cap R\times I_i$ (notice, w is approximate continuous at p). Thus $S_u^1\cap R\times I_i$ isa 1-generalized characteristic in w. This is contradictory to the fact that S_w^1 is an unique 1-contact discontinuity in w.

2° $t\in\tilde{E}_0\subset E_0, x(t)=y(t)$. It follows froms Green's theorem for measure [3] that

$$\gamma(u,w)\{S_w^1\cap\tilde{E}_0\}=\int_{\tilde{E}_0}\tau[\alpha]-[\beta]dt$$
$$=\int_{\tilde{E}_0}-\tau\alpha(u_r,w_r)+\beta(u_r,w_r)dt.$$

Notice that both u_r and w_r lie on the 1-contact curve through $u_l=w_l$. For concreteness, suppose $u_r\in I(w_l,w_r)$, where $I(a,b)$ denote the closed segment with ends a and b. (The contact curve through a given point in u-plane is a straight line.)

We denote

$$v_l\equiv u_r,\quad v_r\equiv w_r,$$

v_l and v_r then can be regarded as two states connected by an 1-contact with speed τ. we have

$$\gamma(u,w)\{S_w^1\cap R\times\tilde{E}_0\}=\int_{\tilde{E}_0}-\tau\alpha(v_l,v_r)+\beta(v_l,v_r)dt$$
$$=\int_{\tilde{E}_0}-d(\tau,v_l,v_r)dt.$$

It is easy to show that

$$d(\tau,v_l,v_r)=0,$$

if v_l and v_r are connected by a contact with speed τ. Thus we have

$$\gamma(u,w)\{R\times\tilde{E}_0\}=\gamma(u,w)\{S_w^{lc}\cap R\times\tilde{E}_0\}$$

$$= \theta_u\{S_w^{1c} \cap R \times \tilde{E}_0\} - \iint_{S_w^{1c} \cap R \times \tilde{E}} \nabla^2 \eta(w) Q f w_x$$

$$\leqslant \text{const} \iint_{R \times \tilde{E}_0} |u - w|^2 dx dt,$$

where $Qf = f(u) - f(w) - \nabla f(w)(u - w)$.

3° $t \in J_i \subset E_2, x(t) > y(t)$. Then $u_l(p) = u_r(p) = w_l(p)$, where $p \in S_w^1 \cap R \times J_i$. Thus

$$\gamma(u, w)\{S_w^1 \cap R \times J_i\} = \int_{J_i} -\tau a(w_l, w_r) + \beta(w_l, w_r) dt = 0.$$

Therefore, we have

$$\gamma(u, w)\{R \times E_2\} \leqslant \text{const} \iint_{R \times E_2} |u - w|^2 dx dt.$$

Sum up results of 1°-3°, we have

$$\gamma(u, w)\{\varphi(t)\} \leqslant \text{const} \iint_{\varphi(t)} |u - w|^2 dx dt, \quad (0 \leqslant t < T). \tag{7}$$

On the other hand, according to the Green's theorem for measure, we have

$$\gamma(u, w)\{\varphi(T)\} = \int_{-\infty}^{\infty} a(u(x,t), w(x,t)) dx$$

$$\geqslant \text{const} \int_{-\infty}^{\infty} |u - w|^2 dx. \tag{8}$$

Here the fact that strictly convex entropy η certainly exists under the conditions of the theorem[4] is used.

The theorem follows immediately by applying Gronwall's inequality to (7) and (8).

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