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The Remainder Representation of Hermite-Birkhoff Interpolation*

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Wang Xinghua[1] has given out a useful tool that represent the kth ($k=0,\cdots,n$) derivatives $R_n^{(k)}(x)$ of the remainder $R_n(x)=f(x)-H_n(x)$ of the Lagrange-Hermite interpolation polynomial $H_n(x)$ of degree n for a function $f(x)$ by the divided differences of order $n+1$ of $f(x)$. This tool has been applied to the estimation of remainders of interpolation splines[9-11] much more conveniently and easily than that one of Green functions or Peano kernels[6,7].

In practice, we often have still to study remainders of a lot of Hermite-Birkhoff interpolation polynomials[2-5]. In this note, our purpose is to extend the remainder formulae of [1] to that Hermite-Birkhoff interpolation polynomials. We shall also represent the kth ($k=0,\cdots,n$) derivatives $R_n^{(k)}(x)$ of the remainder $R_n(x)=f(x)-H_n(x)$ of the Hermite-Birkhoff interpolation polynomial $H_n(x)$ of degree n for a function $f(x)$ by the divided differences of order $n+1$ of $f(x)$. This tool has also been applied to the estimation of approximation degrees of interpolation splines[9-11].

Let $H_n(x)$ be Hermite-Birkhoff interpolation polynomial of degree n for a function $f(x)$, $0\leqslant p-1\leqslant n$, $a_0\leqslant a_1\leqslant\cdots\leqslant a_{p-1}$ and

$$H_n^{(\alpha_i)}(a_i)=f^{(\alpha_i)}(a_i)\qquad(i=0,\cdots,n)\tag{1}$$

where

$$\alpha_i=\max\{j: a_i=a_{i-j}\}\ \ (i=0,\cdots,p-1)$$

and

$$0\leqslant\alpha_i\leqslant n\qquad(i=p,\cdots,n).$$

Specifically, if $p=n+1$, then $H_n(x)$ is just a Lagrange-Hermite interpolation polynomial.

Write $R_n(x)=f(x)-H_n(x)$. We have known the following theorems A and B.

Theorem A[1] If $0\leqslant m\leqslant k\leqslant p-1=n$, then

$$R_n^{(k)}(x)=k!\sum_{\nu=0}^{m-1}f[\overbrace{x,\cdots,x}^{\nu+1},a_0,\cdots,a_n]\frac{\omega_{n+1}^{(k-\nu)}(x)}{(k-\nu)!}$$

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$$+k!\sum_{\nu=m}^{k}f[\overbrace{x,\cdots x,}^{\nu+1},a_0,\cdots,a_{n-\nu+m}]\frac{\omega_{n-\nu+m}^{(k-\nu)}(x)}{(k-\nu)!}(x-a_{n-\nu+m})$$

whenever the above divided differences exist, where

$$\omega_i(x)=\prod_{j=0}^{i-1}(x-a_j)\quad(i=0,\cdots,n+1).$$

Theorem B (see[5] or [8]) If and only if

$$\det[a_i^{\nu-\alpha_i}/(\nu-\alpha_i)!]_{i,\nu=0}^{n}\neq 0,$$

the above Hermite-Birkhoff interpolation polynomial $H_n(x)$ is existent and unique.

From theorem B we may obtain the following lemma.

Lemma If and only if $\Delta\neq 0$, the above Hermite-Birkhoff interpolation polynomial $H_n(x)$ is existent and unique, where*)

$$\Delta=\frac{\partial^{\alpha_p}}{\partial a_p^{\alpha_p}}\frac{\partial^{\alpha_{p+1}}}{\partial a_{p+1}^{\alpha_{p+1}}}\cdots\frac{\partial^{\alpha_n}}{\partial a_n^{\alpha_n}}\prod_{i=p}^{n}\omega_i(a_i).$$

For $x\overline{\in}\{a_0,\cdots,a_n\}$, let

$$I(y)=R_n(y)\omega_p(x)-\omega_p(y)\sum_{\nu=0}^{n-p}C_\nu(x)(y-x)^\nu/\nu!,\tag{2}$$

where

$$C_\nu(x)=\omega_p^{-\nu}(x)\begin{vmatrix}\omega_p(x)&0&0&\cdots&0&R_n(x)\\ \omega_p'(x)&\omega_p(x)&0&\cdots&0&R_n'(x)\\ \omega_p''(x)&2\omega_p'(x)&\omega_p(x)&\cdots&0&R_n''(x)\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ \omega_p^{(\nu-1)}(x)&\binom{\nu-1}{1}\omega_p^{(\nu-2)}(x)&\binom{\nu-1}{2}\omega_p^{(\nu-3)}(x)\cdots&&\omega_p(x)&R_n^{(\nu-1)}(x)\\ \omega_p^{(\nu)}(x)&\binom{\nu}{1}\omega_p^{(\nu-1)}(x)&\binom{\nu}{2}\omega_p^{(\nu-2)}(x)\cdots&&\binom{\nu}{\nu-1}\omega_p'(x)&R_n^{(\nu)}(x)\end{vmatrix}\tag{3}$$

$(\nu=0,\cdots,n-p)$ satisiy

$$I^{(i)}(x)=R_n^{(i)}(x)\omega_p(x)-\sum_{\nu=0}^{i}\binom{i}{\nu}\omega_p^{(i-\nu)}(x)C_\nu(x)=0\quad(i=0,\cdots,n-p).$$

Hence $a_0,\cdots,a_{p-1},\overbrace{x,\cdots,x}^{n-p+1}$ are $n+1$ zeros of $I(y)$. Writing

$$b_i=\begin{cases}a_i & (i=0,\cdots,p-1)\\ x & (i=p,p+1,\cdots)\end{cases}$$

and

$$\theta_i(y)=\prod_{\nu=0}^{i-1}(y-b_\nu)\quad(i=0,1,\cdots),$$

from theorem A, for $k\in\{0,\cdots,n\}$ we have

$$I^{(k)}(y)=k!\omega_p(x)\sum_{\nu=0}^{k}f[\overbrace{y,\cdots,y}^{\nu+1},b_0,\cdots,b_{n-\nu}]\frac{\theta_{n-\nu}^{(k-\nu)}(y)}{(k-\nu)!}(y-b_{n-\nu}).\tag{4}$$

Combining Eqs. (1), (2) and (4), we obtain

$$-I^{(\alpha_i)}(a_i)=\sum_{\nu=0}^{n-p}\frac{C_\nu(x)}{\nu!}\frac{\partial^{\alpha_i}}{\partial a_i^{\alpha_i}}[\omega_p(a_i)(a_i-x)^\nu]\quad(i=p,\cdots,n),\tag{5}$$

where

$$I^{(\alpha_i)}(a_i)=\alpha_i!\omega_p(x)\sum_{\nu=0}^{\alpha_i}f[\overbrace{a_i,\cdots,a_i}^{\nu+1},b_0,\cdots,b_{n-\nu}]\frac{\theta_{n-\nu}^{(\alpha_i-\nu)}(a_i)}{(\alpha_i-\nu)!}(a_i-b_{n-\nu})$$
$$(i=p,\cdots,n).\tag{6}$$

*) In this note the partial derivative is one in which $a_p,\cdots,a_n$ are independent variables.

If $C_\nu(x)/\nu!$ $(\nu=0,\cdots,n-p)$ in Eqs. (5) are regarded as unknown numbers, then the coefficient determinant of Eqs. (5) is

$$\det\left\{\frac{\partial^{\alpha_i}}{\partial a_i^{\alpha_i}}[\omega_p(a_i)(a_i-x)^{\nu-p}]\right\}_{i,\nu=p}^{n}$$

$$=\frac{\partial^{\alpha_p}}{\partial a_p^{\alpha_p}}\frac{\partial^{\alpha_{p+1}}}{\partial a_{p+1}^{\alpha_{p+1}}}\cdots\frac{\partial^{\alpha_n}}{\partial a_n^{\alpha_n}}\det\{\omega_p(a_i)(a_i-x)^{\nu-p}\}_{i,\nu=p}^{n}, \tag{7}$$

where

$$\det\{\omega_p(a_i)(a_i-x)^{\nu-p}\}_{i,\nu=p}^{n}=\det\{\omega_p(a_i)a_i^{\nu-p}\}_{i,\nu=p}^{n}$$
$$=\det\{a_i^{\nu-q}\}_{i,\nu=p}^{n}\prod_{i=p}^{n}\omega_p(a_i)=\prod_{i=p}^{n}\omega_i(a_i).$$

Therefore, the coefficient determinant of Eqs. (5) is just Δ in the above lemma. Hence we have

$$C_\nu(x)=\nu!\,A_\nu/\Delta\ (\nu=0,\cdots,n-p), \tag{8}$$

where A_ν is the determinant whose $(\nu+1)$th column is

$$[-I^{(\alpha_p)}(a_p),\ -I^{(\alpha_{p+1})}(a_{(p+1)}),\cdots,-I^{(\alpha_n)}(a_n)]^T$$

and whose others are the same as the determinant

$$\det\{\omega_p(a_i)(a_i-x)^{\nu-p}\}_{i,\nu=p}^{n}.$$

In virtue of Eqs. (3) and (8), we have the recursive formulae $R_n(x)=C_0(x)$ and

$$R_n^{(k)}(x)=C_k(x)-\omega_p^{-k}(x)\begin{vmatrix}\omega_p(x) & 0 & 0 & \cdots & 0 & R_n(x)\\ \omega_p'(x) & \omega_p(x) & 0 & \cdots & 0 & R_n'(x)\\ \omega_p''(x) & 2\omega_p'(x) & \omega_p(x) & \cdots & 0 & R_n''(x)\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \omega_p^{(k-1)}(x) & \binom{k-1}{1}\omega_p^{(k-2)}(x) & \binom{-1}{k\,2}\omega_p^{(k-3)}(x) & \cdots & \omega_p(x) & R_n^{(k-1)}(x)\\ \omega_p^{(p)}(x) & \binom{k}{1}\omega_p^{(k-1)}(x) & \binom{k}{2}\omega_p^{(k-2)}(x) & \cdots & \binom{k}{k-1}\omega_p'(x) & 0\end{vmatrix}$$

$$(k=0,\cdots,n-p). \tag{9}$$

For $k\in\{n-p+1,\cdots,n\}$, noticing

$$R_n[\overbrace{x,\cdots,x}^{\nu+1},a_0,\cdots,a_{p-1}]=\frac{d^\nu}{dx^\nu}R_n[x,a_0,\cdots,a_{p-1}]/\nu!=\frac{d^\nu}{dx^\nu}\left[\frac{R_n(x)}{\omega_p(x)}\right]\Big/\nu!$$

and using the above theorem A $(m=n-p+1)$, we may obtain

$$R_n^{(k)}(x)=k!\sum_{\nu=0}^{n-p}R_n[\overbrace{x,\cdots,x}^{\nu+1},a_0,\cdots,a_{p-1}]\frac{\omega_p^{(k-\nu)}(x)}{(k-\nu)!}$$

$$+k!\sum_{\nu=n-p+1}^{k}f[\overbrace{x,\cdots,x}^{\nu+1},a_0,\cdots,a_{n-\nu}]\frac{\omega_{n-\nu}^{(k-\nu)}(x)}{(k-\nu)!}(x-a_{n-\nu})$$

$$=k!\sum_{\nu=0}^{n-p}\frac{d^\nu}{dx^\nu}\left[\frac{R_n(x)}{\omega_p(x)}\right]\cdot\frac{\omega_p^{(k-\nu)}(x)}{(k-\nu)!\,\nu!}$$

$$+k!\sum_{\nu=n-p+1}^{k}f[\overbrace{x,\cdots,x}^{\nu+1},a_0,\cdots,a_{n-\nu}]\frac{\omega_{n-\nu}^{(k-\nu)}(x)}{(k-\nu)!}(x-a_{n-\nu})$$

$$(k=n-p+1,\cdots,n). \tag{10}$$

Here $R_n^{(k)}(x)$ and $C_\nu(x)$ in Eqs. (8) ~ (10) have been represented by divided differences of order $n+1$ of $f(x)$ because $I^{(\alpha_i)}(a_i)$ $(i=p,\cdots,n)$ in Eq. (6) have been done. Summing up the aboeve, we have the following theorem.

Theorem If $0\leqslant k\leqslant n$ and $\Delta\neq 0$, then the kth $(k=0,\cdots,n)$ derivatives $R_n^{(k)}(x)$ of the Hermite-Birkhoff interpolation remainder of $f(x)$ may be represented in Eqs. (9) and (10) by divided differences of order $n+1$ of $f(x)$ whenever these divided differences and their derivatives exist.

Finally, if $p=n$, then $\Delta\neq 0$ becomes $\omega_n^{(r)}(a_n)\neq 0$ and the Eqs. (9) and (10) become the simpler forms[9]

$$R_n^{(k)}(x)=k!\sum_{\nu=0}^{k} f[\overbrace{x,\cdots,x}^{\nu+1},a_0,\cdots,a_{n-\nu}]\,\frac{\omega_{n-\nu}^{(k-\nu)}(x)}{(k-\nu)!}\,(x-a_{n-\nu})$$

$$-r!\,\omega_n^{(k)}(x)\sum_{\nu=1}^{r} f[\overbrace{a_n,\cdots,a_n}^{\nu+1},a_0,\cdots,a_{n-\nu}]\,\frac{\omega_{n-\nu}^{(r-\nu)}(a_n)\,(a_n-a_{n-\nu})}{(r-\nu)!\,\omega_n^{(r)}(a_n)}$$

$$(k=0,\cdots,n) \tag{11}$$

whenever these divided differences exist, where $r=\alpha_n$.

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Hermite-Birkhoff 插值的余项表示

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提　　要

本文将王兴华在[1]中给出的 Lagrange-Hermite 插值的余项表示推广到了 Hermite-Birkhoff插值的情形[2-5]。这些工具已有效地用来估计多种插值样条的逼近度[9-11]。