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Using Stopping Point to Describe Stopping Line*

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The stopping line and the stopping point are very important concepts. However few relations between the two concepts have been proved. In this short article, we manage to recount some relations between the two concepts. We also obtain another result. We know that if $\{\lambda_i\}_{i\in\Lambda}$ be a family of stopping line, and Λ is a countable index set, $\bigvee_{i\in\Lambda}\lambda_i$ must be a stopping line; if Λ is not a countable index set, what is the $\bigvee_{i\in\Lambda}\lambda_i$? Here we will point out that $\bigvee_{i\in\Lambda}\lambda_i$ is a stopping line too.

At first, we introduce some definitions and notations.

Let $(\Omega, \mathscr{F})$ be a measurable space, R_+^2 be a positive real plane. For any $t=(t_1,t_2)$, $s=(s_1,s_2)\in R_+^2$, $t\le s$ iff $t_1\le s_1$, $t_2\le s_2$, $t<s$, iff $t_1<s_1$, $t_2<s_2$. Let $\{\mathscr{F}_t\}$ be an increasing family of σ-field, $\mathscr{F}_t\subset\mathscr{F}$ for all $t\in R_+^2$, and $\{\mathscr{F}_t\}$ is right continuous. For all $t\in R_+^2$, let $R_t=\{s|\ s\le t\}$. and $\underline{t}=\partial R_t$, $\bar{t}=\partial\{s|s\geqslant t\}$. We will call the curve G as a separate line, if, for all $t\in G$, $\underline{t}\leqslant G$, that is $R_t<R_G$, where R_G is a region encircled by X axis, Y axis and G. We write S as a family of all separate lines.

Definition 1 If mapping λ: $\Omega\longrightarrow S$ satisfies that for $\forall t\in R_+^2$, $\{\omega|\underline{t}\le\lambda(\omega)\}\in\mathscr{F}_t$, we call λ a stopping line.

Definition 2 If a random vector T: $\Omega\longrightarrow R_+^2$ satisfies that for $\forall\ t\in R_+^2$, $\{\omega|\ T\le t\}\in\mathscr{F}_t$, we call T a stopping point.

Theorem 1 Let λ_1, λ_2 be stopping lines which cannot be comparrble for every ω. If λ_1 and λ_2 have a unique point of intersection (for every ω fixed), the point of intersection must be a stopping point.

Proof Let T be the unique point of intersection of λ_1 and λ_2. For $\forall t\in R_+^2$, we have

$$(T<t)=(\lambda_1\wedge\lambda_2\wedge\underline{t}\ne\lambda_1\wedge\underline{t})\cap(\lambda_1\wedge\lambda_2\wedge\underline{t}\ne\lambda_2\wedge\underline{t}).$$

By[2],

$$(\lambda_1\wedge\lambda_2\wedge\underline{t}\ne\lambda_1\wedge\underline{t})^c=(\lambda_1\wedge\lambda_2\wedge\underline{t}=\lambda_1\wedge\underline{t})\in\mathscr{F}_{\lambda_1\wedge\underline{t}}\subset\mathscr{F}_t,$$ so $(T<t)\in\mathscr{F}_t$, $(T\leqslant t)\in\mathscr{F}_{t+}=\mathscr{F}_t$.

That implies that T is a stopping point. ∎

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However, few stopping lines which satisfy the condition of theorem I, and the theorem 1 does not describe the relation between the stopping point and stopping line well. For this reason, we give an interesting theorem as follows:

Theorem 2 Let λ be a stopping line. For any $\frac{\pi}{2}>\theta>0$, the point of intersection of λ with $y=xtg\,\theta$ must be a stopping point (according to the property of stopping line, the point of intersection of λ with $y=x\,tg\,\theta$ exists and is unique).

Proof At first, we prove that $\|\lambda(\theta)\|$ is $\mathscr{F}$ – measureable ($\|(x,y)\|=\sqrt{x^2+y^2}$), where $\lambda(\theta)$ is the point of intersection of λ with $y=x$ tg θ. For $\forall a\in R_+$

$(\|\lambda(\theta)\|\geqslant a)=(\lambda\geqslant(a\cos\theta,\ a\sin\theta))\in\mathscr{F}_{(a\cos\theta,a\sin\theta)}\subset\mathscr{F}$ so $\|\lambda(\theta)\|$ is $\mathscr{F}$ – measureable.

Now, we begin to prove the Theorem. For fixed θ, let

$$S_n(\theta)=\sum_{n=1}^{\infty}\left(\frac{k}{2^n}\cos\theta,\frac{k}{2^n}\sin\theta\right)I_{\left(\frac{k-1}{2^n}\leqslant\|\lambda(\theta)\|<\frac{k}{2^n}\right)}.$$

For $\forall t_0\in R_+^2$, by the definition of $S_n(\theta)$, $\exists k_o$ satisfies the following relation

$$\frac{k_o}{2^n}\cos\theta\leqslant\|t_o\|\cos\theta_1,$$

$$\frac{k_o}{2^n}\sin\theta\leqslant\|t_o\|\ \sin\theta_1,$$

where $\theta_1=<\vec{t_o},\overrightarrow{(1,0)}>$ denotes the angle between $\vec{t_o}$ and $\overrightarrow{(1,0)}$, and

$$(S_n(\theta)\leq t_o)=(0\leq\|\lambda(\theta)\|<\frac{k_o}{2^n}),$$

hence that $(S_n(\theta)\leq t_o)=(\|\lambda(\theta)\|<\frac{k_o}{2^n})\in\mathscr{F}_{(\frac{k_o}{2^n}\cos\theta,\ \frac{k_o}{2^n}\sin\theta)}\subseteq\mathscr{F}_{t_o}$ that is $S_n(\theta)$ is a stopping point. By the definition of $S_n(\theta)$, we know that $S_n(\theta)\downarrow\lambda(\theta)$, so $\lambda(\theta)$ is a stopping point. ∎

From the proof of the Theorem 2, it is clear that, if λ and X axis have a unique point of intersection, the Theorem 2 is also truth for $\theta=0$; if λ and Y axis have a unique point of intersection, so is the Theorem 2 for $\theta=\frac{\pi}{2}$.

The following theorem will give the relation between stopping point and stopping line.

Theorem 3 Let λ be a random curve, it has a unique point of intersection with X axis and Y axis respectively, λ is a stopping line iff, for every ω fixed, $\lambda(\omega)$ is a separate line, and, for every θ fixed, the unique point of intersection of λ with $y=x$ tg θ is a stopping point,

Proof By the Theorem 2, the "only if" is clear. Now we prove the "if",

At first, for $\forall\ t\in\ R_+^2$, we prove

$$(*) \qquad (\lambda \geq t)^c = \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} (\lambda_{mn} < t),$$

where $\lambda_{mn} = \lambda\left(\frac{m}{2^n}, \frac{\pi}{2}\right)$, $(m \leq 2^n)$. By the condition of the Theorem, we know that $\lambda(\omega)$ is a separate line, for every ω fixed. If $\omega \in (\lambda \geq t)^c$. that is, $\lambda(\omega) \not\geq t$, so $R_t^0 \cap (R_{\lambda(\omega)}^c)^0 \neq \emptyset$. There must exist some $\lambda_{m,n}$, such that $\lambda_{m,n}$, $(\omega) \in \partial R_{\lambda(\omega)}$, and $\lambda_{m,n}$, $(\omega) < t$, hence that

$$(\lambda \geq t)^c \subset \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} (\lambda_{mn} < t);$$

The opposite direction is clear, so $(*)$ is truth.

As $\{\lambda_{mn}\}$ are stopping points, we know that

$$(\lambda \geq t)^c \in \mathscr{F}_t,$$

that is, λ is a stopping line. ∎

Let $\{\lambda_a\}_{a \in \Lambda}$ be a family of stopping line, where Λ may be an uncountable index set. We define

$$\lambda(\theta) = (\bigvee_{a \in \Lambda} \lambda_a)(\theta) = \text{ess. sup } \lambda_a(\theta).$$

Theorem 4 Let $\{\lambda_a\}_{a \in \Lambda}$ be a family of stopping line, where Λ may be an uncountable index set, and every λ_a has a unique point of intersection with X axis and Y axis respectively(this condition is not necessary, it can be removed, by seeing the remark (2) following the proof of this Theorem). Let $\lambda = \bigvee_{a \in \Lambda} \lambda_a$ then λ is a stopping line.

Proof Let $\theta_{m,n} = \frac{m}{2^n} \cdot \frac{\pi}{2}$, $m,n = 1,2,\cdots\cdots, m \leq 2^n$. For every $\theta_{m,n}$, $\{\lambda_a(\theta_{m,n})\}$ is a no empty family of stopping point, and $\lambda(\theta_{m,n}) = \bigvee_{a \in \Lambda} \lambda_a(\theta_{m,n}) = \text{ess. sup}_{a \in \Lambda}$ $\{\lambda_a(\theta_{m,n})\}$. By [1]1.23, there exists countable elements of $\{\lambda_a(\theta_{m,n})\}$, suppose which are $\lambda_{a_k^{m,n}}(\theta_{m,n})$, $a_k^{m,n} \in \Lambda_{m,n}$, $\Lambda_{m,n}$ is a countable index set, such that

$$\lambda(\theta_{m,n}) = \bigvee_{a_k^{m,n} \in \Lambda_{m,n}} \lambda_{a_k^{m,n}}(\theta_{m,n}).$$

Now we obtain a family of stopping line $\{\lambda_{a_k^{m,n}}\}, a_k^{m,n} \in \Lambda_{m,n}$, m, $n = 1, 2, \cdots\cdots$. For convenience, we mark $\{\lambda_{a_k^{m,n}}\}$, $a_k^{m,n} \in \Lambda_{m,n}, m,n = 1, 2\cdots\cdots$, as $\{\lambda_k\}$, $k = 1, 2, \cdots\cdots$. Now; it remains to prove $\lambda = \bigvee_k \lambda_k$. If $\bigvee_k \lambda_k \neq \lambda$, there must exist some $\omega_0 \in \Omega$ and $\theta_0 > 0$ (for $\theta_0 = 0$, the proof is similar.) such that

$$\bigvee_k T_k(\theta_0, \omega_0) < T(\theta_0, \omega_0).$$

We can chose a rational number $q = \frac{m}{2}$ such that $\theta_0 > q\frac{\pi}{2} > 0$ and

$$\angle \overrightarrow{\lambda(\theta_0,\omega_0)}, \overrightarrow{\lambda(q\frac{\pi}{2}, \omega_0) - \bigvee_l \lambda_l(\theta_0,\omega_0)} > \angle\theta_0, \text{hence that}$$

$$(**)\qquad \bigvee_l \lambda_l(\theta_0,\omega_0)<\lambda(q\frac{\pi}{2},\omega_0)=\bigvee_l \lambda_l(q\frac{\pi}{2},\omega_0).$$

Since $\bigvee_l \lambda_l$ is a stopping line, $\bigvee_l \lambda_l(\omega_0)$ must be a separate line that contract with $(**)$. So $\bigvee_l \lambda_0=\lambda$, that is, λ is a stopping line. ∎

Theorem 5 Let $\{\lambda_\alpha\}_{\alpha\in\Lambda}$ be a family of stopping line, where Λ may be an uncountable index set, and every λ_α has a unique point of intersection with X axis and Y axis respectively. Let $\lambda=\bigvee_{\alpha\in\Lambda}\lambda_\alpha$, *then* $\lambda=ess.\ sup\ \lambda_\alpha$.

Proof Let $\mathscr{H}=\{\lambda_\alpha,\ \alpha\in\Lambda\}$. By the definition of λ, for all $\lambda_\alpha\in\mathscr{H}$, we have

$$\lambda_\alpha\le\lambda.$$

If there is a stopping line λ', such that

$$\lambda_\alpha\le\lambda'\qquad \text{for all } \lambda_\alpha\in\mathscr{H}.$$

Then, for every $0\leqslant\theta\leqslant\frac{\pi}{2}$ fixed,

$$\lambda_\alpha(\theta)\leqslant\lambda'(\theta).$$

By the definition of $\lambda(\theta)$, the above inequality involves that $\lambda(\theta)\leqslant\lambda'(\theta)$, ao that $\lambda\leqslant\lambda'$. This points out that $\lambda=ess.\ \sup\ \lambda_\alpha$. ∎

Remark (1) Using the same method, we can prove $\bigwedge_{\alpha\in\Lambda}\lambda_\alpha=\lambda$ is a stopping line. (2) If $\{\lambda_\alpha\}_{\alpha\in\Lambda}$ have more than one point of intersection with X axis or Y axis, the Theorem 4 also holds. Let

$$\sup\{\lambda(0)\}\triangleq\lambda^+(0).$$

For $\forall\ t=(t_1,t_2)\in R^2_+$, we have $(\lambda^+(0)\leqslant t)=(|\lambda^+(0)|\leqslant t_1)$, since

$$(|\lambda^+(0)|\ge t_1)=(\lambda\ge(t_1,\ 0))\in\mathscr{F}_{(t_1,0)}\in\mathscr{F}_t,$$

this implies that

$$(\lambda^+(0)<t)\in\mathscr{F}_t.$$

and

$$(\lambda^+(0)\le t)=\bigcap_{n=1}^{\infty}(\lambda^+(0)<t_n)\in\mathscr{F}_{t+}\subset\mathscr{F}_t,$$

where $t_n\downarrow t$, $t_n\in R^2_+$. This points that $\lambda^+(0)$ also is a stoppingpoint. Similarly, $\lambda^+\left(\frac{\pi}{2}\right)=\sup\left\{\lambda(\frac{\pi}{2})\right\}$ is a stopping. $\lambda=\bigvee_{\alpha\in\Lambda}\lambda_\alpha$ contains that $\lambda^+(0)=\bigvee_{\alpha\in\Lambda}\lambda^+(0)$, $\lambda^+\left(\frac{\pi}{2}\right)=\bigvee_{\alpha\in\Lambda}\lambda^+_\alpha(\frac{\pi}{2})$, so we can chose a countable family of stopping line $\{\lambda_k\}$, $k=1,2,\cdots\cdots$, such that

$$\bigvee_k\lambda^+_k(0)=\lambda^+(0),\ \bigvee_k\lambda^+_k(\frac{\pi}{2})=\lambda^+(\frac{\pi}{2}),\ \bigvee_k\lambda_k(\theta_{m,n})=\lambda(\theta_{m,n}),\ \theta_{m,n}=\frac{m}{2^n}\cdot\frac{\pi}{2},$$

$m,\ n=1,2,\cdots\cdots$, We can prove $\lambda=\bigvee_k\lambda_k$ is a stopping line by the similar way of the Thoerem 4.

Reference

[1]. 严加安、鞅与随机积分引论，上海科学技术出版社 (1981).

[2]. Ely Merzbach, Processus Stochastiques a Indices Partillement Ordonnes.