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The Fixed Point Theorem for Some Contraction Mappings*

Y.R. Xu (徐叶润)

(Shanghai College of Traditional Chinese Medicine)

The fixed point theorem has many important application in the theory of differential equations. The book of Kolmogorov and Fomin [9] provides excellent illustration of the use of fixed point theorem in analysis. Various contraction mappings and their fixed point theorem are given in B.E. Rhoades [10].

In the present paper, the problem for some generalized contraction mapping is considered, and some fixed point theorem is obtained.

First of all, we shall use the following notation.

R^+: the set of alll non-negative reel numbers.

R^{+5}: the set of all points $(x_1, x_2, x_3, x_4, x_5)$ with $x_i \in R^+$. $i = 1, \cdots, 5$.

I^+: the set of all positive integers.

Λ: the set of any index.

H: the set of all functions from R^{+5} to R^+ which is upper semicontinuous and nondecreasing about each variable.

Theorem 1 Let (X, d) is complete metric space, $\{S\}_{i\in\Lambda}$ is a family of mappings form X to X. If there exist a function $h \in H$, a mapping T from X onto X and a mapping n_i from X to I^+ for each $i \in \Lambda$, such that:

(1) For all $i \in \Lambda$, $S_i T = T S_i$.

(2) For all $x \in X$, each $i \in \Lambda$, $n_i(S_i x) = n_i(x)$, $n_i(Tx) = n_i(x)$.

(3) When $a = b = 1$ or $a = 0, b = 2$ or $a = 2, b = 0$. the inequality $h(t, t, at, bt, t) < t$ holds for $t > 0$.

(4) For any $x, y \in X$. when $i \neq j$, the inequality

$$d(S_i^{n_i(x)}(x),\ S_j^{n_j(y)}(y)) \leqslant h\{d(Tx, Ty),\ d(Tx, S_i^{n_i(x)}(x)).\ d(Tx, S_j^{n_j(y)}(y)),\ d(Ty, S_i^{n_i(x)}(x)),\ d(Ty, S_j^{n_j(y)}(y))\}.$$

holds.

Then $\{S_i\}_{i\in\Lambda}$ and T have a unique common fixed point.

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Proof For fixed i, $j\in\Lambda$. Let $x_0\in X$. Consider the series $\{X_n\}$ defined by

$$Tx_{2n+1}=S_i^{n_i(x,n)}(x_{2n}) \qquad Tx_{2n+2}=S_j^{n_j(x_{2n+1})}(x_{2n+1}) \tag{1}$$

This series is well-defined since $TX=X$ and not unique. The proof is divided into four parts.

I. $d(TX_{m+1},\ TX_m)\to 0$, as $m\to\infty$.

If we let $a_{2n+2}=d(TX_{2n+2},\ TX_{2n+1})$, $a_{2n+1}=d(TX_{2n+1},\ TX_{2n})$. We have

$$\begin{aligned} a_{2n+2}&\leqslant h(a_{2n+1},\ a_{2n+1},\ a_{2n+2}+a_{2n+1},\ 0,\ a_{2n+2}),\\ a_{2n+1}&\leqslant h(a_{2n},\ a_{2n},\ a_{2n}+a_{2n+1},\ 0,\ a_{2n+2}), \end{aligned} \tag{2}$$

since (1) and condition (4). Hence

$$0\leqslant a_{2n+2}<a_{2n+1}<a_{2n}<\cdots<a_1,$$

$\lim\limits_{n\to\infty}a_n$ exist and denote by a. It is easy from (2) that $a=0$, i.e.

$$d(TX_{m+1},\ TX_m)\to 0,\ \text{as } m\to\infty.$$

II. $\{TX_n\}$ is a Cauchy series.

It is sufficient to prove $d(TX_{2n+p},\ TX_{2n})\to 0$, $d(TX_{2n+1+p},\ TX_{2n+1})\to 0$, as $n\to\infty$, for every $p\in I^+$. We only prove the first term. The proof of second term is similar.

If p is even. For any given $\varepsilon>0$, when n as sufficient large, we have $d(TX_{2n+1},\ TX_{2n})<\varepsilon$, $d(TX_{2n+p},\ TX_{2n+p+1})<\varepsilon$. Since $d(TX_{m+1},\ TX_m)\to 0$ as $m\to\infty$, so

$$\begin{aligned} d(TX_{2n+p},\ TX_{2n})&\leqslant d(S_i^{n_i(x_{2n})}(x_{2n}),\ S_j^{n_j(x_{2n+p-1})}(x_{2n+p-1}))+d(TX_{2n+1},\ TX_{2n})\\ &\leqslant h(\varepsilon+d(TX_{2n+p},\ TX_{2n}),\ \varepsilon,\ (d(TX_{2n+p},TX_{2n}),d(TX_{2n+p},TX_{2n})\\ &\quad +2\varepsilon,\varepsilon)+d(TX_{2n+1},\ TX_{2n})\\ &\leqslant h(\varepsilon+c_p,\ \varepsilon,\ c_p,\ c_p+2\varepsilon,\ \varepsilon)+\varepsilon, \end{aligned}$$

where $c_p=\limsup\limits_{n\to\infty} d(TX_{2n+p},\ TX_{2n})$. Furthermore we have

$$c_p\leqslant h(\varepsilon+c_p,\ \varepsilon,\ c_p,\ c_p+2\varepsilon,\ \varepsilon)+\varepsilon.$$

Let ε monotone decrease to 0. We have

$$c_p<h(c_p,\ 0,\ c_p,\ c_p\ 0)\leqslant h(c_p,\ c_p,\ c_p,\ c_p,\ c_p).$$

But, for fixed p, $c_p<\infty$ because of $d(TX_{2n+p},\ TX_{2n})<p\varepsilon$. Therefore we have $c_p=0$, i.e. $d(TX_{2n+p},\ TX_{2n})\to 0$, as $n\to\infty$ and p is even.

If p is odd. In this case, we have

$$d(TX_{2n+p},TX_{2n})\leqslant d(TX_{2n+p}.TX_{2n+2})+d(TX_{2n+2},TX_{2n+1})+d(TX_{2n+1},TX_{2n}).$$

The next proof is similar to the above proof.

Taken altogether, $\{TX_n\}$ is a Cauchy series in X. Since X is complete, $TX_n\to Z_{ij}$ for some $Z_{ij}\in X$. T is onto, so there exist $X_{ij}\in X$ such that $TX_{ij}=Z_{ij}$.

III. T, S_i, S_i have an unique common fixed point.

First we prove $z_{ij}=TX_{ij}=S_i^{n_i(x_{ij})}(x_{ij})=S_j^{n_j(x_{ij})}(x_{ij})$. In fact, if $c=d(z_{ij},S_i^{n_i(x_{ij})}x_{ij}))>0$. Then $TX_n\to Z_{ij}$, as $n\to\infty$, from which it follows that $S_j^{n_j(x_{2n+1})}(x_{2n+1})\to z_{ij}$. Since $d(TX_{2n+2}TX_{2n+1})\to 0$, for any given $\varepsilon<0$, there exist $N=N(\varepsilon)>0$ such that

$$d(TX_{2n+2},\ TX_{2n+1})<\varepsilon,\ d(TX_{2n+2},\ z_{ij})<\varepsilon,\ d(s^{n_j(x_{2n+1})}(x_{2n+1}),\ z_{ij})<\varepsilon$$

for $n>N$. On the other hand,

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$$c \leqslant d(S_i^{n_i(x_{ij})}(x_{ij}),\ S_j^{n_j(x_{2n+1})}(x_{2n+1})) + d(S_j^{n_j(x_{2n+1})}(x_{2n+1}),\ z_{ij})$$
$$\leqslant h\{d(z_{ij}, TX_{2n+1}),\ d(z_{ij}, S_j^{n_j(x_{ij})}(x_{ij})),\ d(z_{ij}, TX_{2n+1}),\ d(TX_{2n+2}, z_{ij}) + d(z_{ij}, S_i^{n_i(x_{ij})}$$
$$\cdot (x_{ij})),\ d(TX_{2n+2}, TX_{2n+1})\} + d(S_j^{n_j(x_{2n+1})}x_{2n+1}, z_{ij})$$
$$\leqslant h(\varepsilon,\ c,\ \varepsilon,\ \varepsilon + c,\ \varepsilon) + \varepsilon.$$

Let ε monotone decrease to 0. We have $c < h\ (0, c, 0, c, 0) < c$, a contradiction. Hence $z_{ij} = S_j^{n_i(x_{ij})}(x_{ij})$ The proof of $z_{ij} = S_j^{n_j(x_{ij})}(x_{ij})$ is similar.

Next we prove that z_{ij} is common fixed point of T, S_i, S_j. It is knovn from above that $S_i z_{ij} = S_j^{n_i(x_{ij})}(S_i x_{ij}) = S_j^{n_i(S_i x_{ij})}(S_i x_{ij})$ exist. Therefor

$$d(z_{ij}, S_i z_{ij}) = d(S_j^{n_i(S_i x_{ij})}(S_i x_{ij}),\ S_j^{n_j(x_{ij})}(x_{ij}))$$
$$\leqslant h\{d(z_{ij}, S_i z_{ij}),\ 0,\ d(z_{ij}, S_i z_{ij}),\ d(z_{ij}, S_i z_{ij}),\ 0\}.$$

It follow that $S_i Z_{ij} = Z_{ij}$. With the similar proof, $S_j Z_{ij} = Z_{ij}$. The inequality

$$d(z_{ij},\ Tz_{ij}) = d(S_j^{n_i(x_{ij})}(x_{ij}),\ S_j^{n_j(Tx_{ij})}(Tx_{ij}))$$
$$\leqslant h\{d(z_{ij},\ Tz_{ij}),\ 0,\ d(z_{ij},\ Tz_{ij}),\ d(z_{ij},\ Tz_{ij}),\ 0\}$$

implies that $TZ_{ij} = z_{ij}$. Thus z_{ij} is common fixed point of T, S_i, S_j.

Finally we prove the common fixed point of T, S_i, S_j is unique. Suppose there are two distinct common fixed point z_1, z_2. Then $d(z_1, z_2) > 0$, and $z_l = S_i z_l = S_j z_l = Tz_l$ $i = 1, 2$. Therefor

$$0 < d(z_1,\ z_2) = d(S_i^{n_i(z_1)}(z_1),\ S_j^{n_j(z_1)}(z_2) \leqslant h\{d(z_1, z_2),\ 0,\ d(z_1, z_2),$$
$$d(z_1, z_2),\ 0\} < d(z_1, z_2)$$

which is impossible. So the common fixed point of T, S_i, S_j is unique.

IV. $\{S_i\}_{i \in \Lambda}$ and T have a unique common fixed point.

It folllows from III that any S_i, S_j and T have a unique common fixed point z_{ij}. We shall prove that all $\{z_{ij}\}$ are same.

If there exist common fixed point z_{ij}, z_{lk} such that $z_{ij} \neq z_{kl}$. Then there is, at least, a distinct index. Assume $i \neq k$. We have

$$0 < d(z_{ij},\ z_{kl}) = d(S_i^{n_i(x_{ij})}(z_{ij}),\ S_k^{n_k(z_{kl})}(z_{kl})) \leqslant h(d(z_{ij}, z_{kl}),\ 0,\ d(z_{ij}, z_{kl}),$$
$$d(z_{ij}, z_{kl}), 0) < d(z_{ij}, z_{kl})$$

which is impossible. So all z_{ij} are same. $\{S_i\}_{i \in \Lambda}$ and T have a unique common fixed point.

It is easy to reduce from theorem I that

Theorem 2 If the condition (4) in the theorem is replaced by (4'): For all $x, y \in X$, $i, j \in \Lambda$, $i \neq j$.

$$d(S_i^{n_i(x)}(x), S_j^{n_j(y)}(y)) < h\{d(T^r x, T^r y),\ d(T^r x, S_i^{n_i}(x)),\ d(T^r x, S_j^{n_i(y)}(y))$$
$$d(T^r y, S_i^{n_i(x)}(x)),\ d(T^r y, S_j^{n_j(y)}(y))\}$$

where r is some positive integer.

Then $\{S_i\}_{i \in \Lambda}$ and T have a unique fixed point.

Corollary: If $(X,\ d)$ and $\{S_i\}_{i \in \Lambda}$, $\{n_i\}_{i \in \Lambda}$ are the same as in theorem 1. For any $x, y \in X$, $\{S_i\}_{i \in \Lambda}$ satisfy the inequality

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$$d(S_i^{n_i(x)}(x), S_j^{n_j(y)}(y)) \leq a_1(d(x,y))d(x,y) + a_2(d(x,y))d(x, S_i^{n_i(x)}(x)) + a_3(d(x,y))d(x, S_j^{n_j(y)}(y)) + a_4(d(x,y))d(y, S_i^{n_i(x)}(x)) + a_5(d(x,y))d(y, S_j^{n_j(y)}(y))$$

where $a_i(t)$ is a function from R^+ to $(0,1)$ and monotone nondecreasing, $\sum_{i=1}^{5} a_i(t) < 1$. Then $\{S_i\}_{i \in \Lambda}$ have a unique common fixed point. In particular, if $a_i(t) = a_i$, for $t \in R^+$, $i = 1, \cdots, 5$. Then we have obtained the fixed point theorem of (B) type contraction mapping of [6].

Remarks 1. Taking T as an identity mapping, $h(t_1, t_2, t_3, t_4, t_5) = \alpha \max(t_1, t_2, t_5, \frac{1}{2}(t_3 + t_4))$, $\alpha \in (0,1)$ yields the result of Ray. B. K. & Rhoades, B. E. [7] i.e. the fixed point theorem of (A) type contraction mapping of [6]

2. If T is taken as an identity mapping, $h(t_1, t_2, t_3, t_4, t_5) = r \max(t_1, t_2, t_5, t_3/2, t_4/2)$. $r \in (0,1)$. Then the result in [8] is obtained.

3. The conclusion is also exist if we replace the condition that "T is onto" by "T is continuous" and take $n_i(x) = p_i$. The proof is different from the proof of theorem. So the result in [1] [2] [3] [4] been obtaiined.

3. The conclusion is also exist if-we replace the condition that "T is onto" by "T is continuous", and take $n_i(x) = p_i$. The proof is different from the proof of theorem. So the result in [1] [2] [3] [4] have been obtained.

4. In this paper, we treat some contraction mapping whioh cann't be solved by the method of Chang and Ding in [5]. For an example, let $h(t_1, t_2, t_3, t_4, t_5) = t_1 t_2 t_5 / (1 + t_2)(1 + t_5)$. Then the common fixed point of a family mapping which satisfy the inequality.

$$d(S_i^{n_i(x)}(x), S_j^{n_j(y)}(y)) \leq d(Tx, Ty) d(Tx, S_i^{n_i(x)}(x)) d(Ty, S_j^{n_j(y)}(y)) / (1 + d(Tx, S_i^{n_i(x)}(u)))(1 + d(Ty, S_i^{n_i(y)}(y)))$$

can't be determined by the method of [5]. But if followed immediately from theorem 1 that all $\{S_i\}_{i \in \Lambda}$ have a unique common fixed point.

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