第5卷 第4期 数学研究与评论 Vol.5 No.4
1985年10月 JOURNAL OF MATHEMATICAL RESEARCH AND EXPOSITION Oct. 1985

The Decision Problems for Properties of Transformation Groups*

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Abstract

In this paper the fundamental result about the decision problems for properties of FRT-groups (i. e. the groups each of which is isomorphic with a group generated by a finite number of recursive transformations) has been proved: Let p be any algebraic property for groups such that there is a FRT-group G_1 which has the property p, a FRT-group G_2 which has not the property p, and G_2 is not isomorphic with any subgroup of any FRT-group which has the property p. Then the problem of deciding, for any given groap P genetated by a finite number of recursive transformations, whether or not the group G isomorphic with P has the property p is unsolvable.

A sequence of important consequences may be obtained from the fundamental result.

Let G be any abstract group. If there exists a transformation group $P=(A,B,\cdots,C)$ generated by a finite number of recursive transformations $A,B,\cdots,C$, such that $G\simeq P$, then we say that P is a representative of G as a group generated by a finite number of recursive transformations, or P is a FRT-representative of G; say that G is a group which has a representative as a group generated by a finite number of recursive transformations, or G is a FRT-group, and write that $G=G_P$.

In this paper the following fundamental result about the decision problems for algebraic properties of FRT-groups is proved.

Theorem Let p be any algebraic property for groups (i. e. a property for groups which is invariable under isomorphisms) such that there is a FRT-group G_1 which has the property p, a FRT-group G_2 which has not the property p, and G_2

* Received Nov. 5, 1983. Recommended by Yang An-zhou (杨安洲).

is not isomorphic with any subgroup of and FRT-group which has the property p. Then the problem of deciding, for any given FRT-representative P, whether or not G_P has the property p is unsolvable.

From this the following important consequences follows.

Corollary Let p be any algebraic property for groups which is hereditary and non-trivial with respect to FRT-groups (i. e. there is a FRT-group G_1 which has the property and a FRT-group G_2 which has not the property). Then the problem of deciding, for any given FRT-representative P, whether or not G_P has the property p is unsolvable.

Thus the following decision problems are all unsolvable: deciding, for any given FRT-representative P, whether or not G_P is

(1) cyclic, (2) finite, (3) free, (4) commutative, (5) solvable,

(6) the group of units, (7) torsion-free group, and so on.

To prove the theorem we need a lemma.

Lemma Let $\mathscr{P}$ be the set of all FRT-representatives, $\mathscr{T}$ be the set of all recursive transformations on N. Then there exists a mapping τ from $\mathscr{P}\times\mathscr{T}$ into $\mathscr{P}$ such that, for any $(P,T)\in\mathscr{P}\times\mathscr{T}$,

$$\tau(P,T)\simeq\begin{cases}\{e\}, & \text{if } T=I;\\ P, & \text{if } T\neq I,\end{cases} \tag{1}$$

where I is the identity transformation.

Proof For any recursive transformation S on N define transformations S_2 and S_3 on $N\times N\times N$ as follows:

S_2: $(i,i,k)\to(i,iS,k)$; $(i,iS,k)\to(i,i,k)$; $(i,j,k)\to(i,j,k)$, if $j\neq i$, iS.

S_3: $(i,i,k)\to(i,i,kS)$; $(i,j,k)\to(i,j,k)$, if $j\neq i$.

For any recursive transformations S and T on N, definetransformations S'_T and S^*_T on $N\times N\times N$ as follows:

$$S'_T=S_3T_2S_3^{-1}T_2.$$

S^*_T: $(i,j,k)\to(i,j,k)S'_T$, if $i=j$, or $i\neq j$ and $(i,j)\bar{\in}T$;

$(i,j,k)\to(i,j,kS^2)S'_T$, if $i\neq j$ and $(i,j)\in T$.

It is easy to see that S_2,S_3,S'_T,S^*_T are all recursive, and that

S'_T: $(i,i,k)\to(i,i,k)$ if $(i,i)\in T$; $(i,i,k)\to(i,i,kS)$ if $(i,i)\bar{\in}T$;

$(i,j,k)\to(i,j,kS^{-1})$ if $i\neq j$ and $(i,j)\in T$; $(i,j,k)\to(i,j,k)$ if $i\neq j$ and $(i,j)\bar{\in}T$.

Therefore S^*_T: $(i,i,k)\to(i,i,k)$ if $(i,i)\in T$; $(i,i,k)\to(i,i,kS)$ if $(i,i)\bar{\in}T$;

$(i,j,k)\to(i,j,kS)$, if $i\neq j$ and $(i,j)\in T$;

$(i,j,k)\to(i,j,k)$, if $i\neq j$ and $(i,j)\bar{\in}T$.

Let f be any recursive correspondence between $N\times N\times N$ onto N, and write the image of $(i,j,k)\in N\times N\times N$ under f to be $\langle i,j,k\rangle\in N$. For any recursive transformations S and T on N define a transformation S_T on N as follows:

$$S_T=f^{-1}S_T^*f, \qquad (2)$$

thus S_T is recursive,

For any FRT-representative $P=(A,B,\cdots,C)$ and any recursive transformation T on N, form FRT-representative P_T as:

$P_T=(A_T,B_T,\cdots,C_T)$. Let τ: $\mathscr{P}\times\mathscr{T}\to\mathscr{P}$ be given by $\tau(P,T)=P_T$, we prove that τ satisfies the property (1).

If $T=I$, then, for any $i,j\in N$, $(i,j)\in T$ iff $i=j$. Hence, by (2), $A_T=B_T=\cdots=C_T=I$, and therefore $P_T\simeq\{e\}$.

If $T\neq I$, then let $\varphi: P\to P_T$ satisfy, for any word $XY\cdots Z\in P$ $(X,Y,\cdots,Z\in\{A, A^{-1},B,B^{-1},\cdots,C,C^{-1}\})$,

$$\varphi:\ XY\cdots Z\to X_TY_T\cdots Z_T,$$

it is clear that φ is a mapping from P onto P_T, and $\langle i,j,k\rangle X_TY_T\cdots Z_T$

$$=\begin{cases}\langle i,j,k\rangle Y_T\cdots Z_T, & \text{if } i=j \text{ and } (i,j)\in T, \text{ or } i\neq j \text{ and } (i,j)\bar{\in} T\\ \langle i,j,kX\rangle Y_T\cdots Z_T, & \text{if } i=j \text{ and } (i,j)\bar{\in} T, \text{ or } i\neq j \text{ and } (i,j)\in T\end{cases}$$

$$=\begin{cases}\langle i,j,k\rangle, & \text{if } i=j \text{ and } (i,j)\in T, \text{ or } i\neq j \text{ and } (i,j)\bar{\in} T\\ \langle i,j,k(XY\cdots Z)\rangle, & \text{if } i=j \text{ and } (i,j)\bar{\in} T, \text{ or } i\neq j \text{ and } (i,j)\in T\end{cases}$$

$$=\langle i,j,k\rangle(XY\cdots Z)_T.$$

Hence P is homomorphic with P_T under φ.

Now, if $A=B=\cdots=C=I$, then from $P\sim P_T$ it follows that $P\simeq P_T(\simeq\{e\})$. Otherwise, if $XY\cdots Z\neq I$, then there is $k\in N$ such that $k\neq k(XY\cdots Z)$.

Since $T\neq I$, hence there are m and n such that $m\neq n$ and $(m,n)\in T$. Thus $\langle m,n,k\rangle X_TY_T\cdots Z_T=\langle m,n,k(XY\cdots Z)\rangle\neq\langle m,n,k\rangle$.

So that $X_TY_T\cdots Z_T\neq I$. Therefore we have $P\simeq P_T$ again.

Now we prove the theorem.

Proof of theorem Let us, for any recursive transformations S and T on N, write the recursive transformationin $\{(a,b)\mid(a=2x\&b=2y\&(x,y)\in S)\vee(a=2x+1\& b=2y+1\&(x,y)\in T)$ on N as $\binom{S}{T}$; for any FRT-representative $P=(R_1,R_2,\cdots,R_m)$ and $Q=(S_1,S_2,\cdots,S_n)$, write the FRT-representative

$$\left(\binom{R_1}{I},\binom{R_2}{I},\cdots,\binom{R_m}{I},\binom{I}{S_1},\binom{I}{S_2},\cdots,\binom{I}{S_n}\right)$$

as $P*Q$.

Let the FRT-representatives of G_1 and G_2 be P_1 and P_2 respectively, and let P_0 be any FRT-representative which has an unsolvable word problem (for existence of P_0 see [1]). Form FRT-representative Q:

$$Q = P_2^* P_0.$$

Thus Q has an unsolvable word problem. For any word w of Q (which is obviously a recursive transformation on N), form FRT-representative $P(w)$:

$$P(w) = P_1^* Q_w.$$

Thus, if $w = I$, then $Q_w \simeq \{e\}$, $P(w) \simeq P_1$ has property p. If $w \neq I$, then $P(w) \simeq P_1^* Q$ (for $Q_w \simeq Q$). Hence P_2 is isomorphic with a subgroup of $P(w)$. By hypotheses, $P(w)$ has not property p.

Now, if the decision problem mentioned in the Theorem is solvable, the problem of deciding, for any $w \in Q$, whether or not $w = I$ is solvable, too. Hence from the fact that Q has an unsolvable word problem, our theorem follows. ∎

Reference

[1] Lin Yucai, Relation Word Problems.