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Some Hypersurfaces with Constant Mean Curvature in a Conformally Flat Riemannian Manifold

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§1. T. Otsuki [1] studied the minimal hypersurface V^n of a Riemannian manifold S^{n+1} of constant curvature if the number of the distinct principal normal curvatures is two and the multiplicities of them are at least two. He proved that V^n is locally the Riemannian prodruct $S^{l_1}\times S^{l_2}$ of two Riemannian manifolds S^{l_1} and S^{l_2} of constant curvature, where l_1 and l_2 are these multiplicities, respectively. In the present paper S^m denotes an m-dimensional Riemannian manifold of constant curvature. In the paper [2] we studied the conformally flat minimal hypersurface M^n of S^{n+1} $(n\geqslant 4)$ and showed that if the scalar curvature of M^n is constant, then either M^n is totally geodesic and, therefore, M^n is S^n, or M^n is locally a Riemannian product $S^{n-1}\times R^1$, where R^1 is a line. The paper [3] generalized this result to the conformally flat hypersurface M^n $(n\geqslant 4)$ with constant mean curvature in S^{n+1} and proved that if the scalar curvature of M^n is constant, then either M^n is S^n, or M^n is locally a Riemannian product $S^{n-1}\times R^1$.

In the present paper we consider the hypersurface V^n with constant mean curvature in a coniormally flat Riemannian manifold M^{n+1} $(n\geqslant 4)$. We shall prove that if V^n is conformally flat and the normal direction of V^n is a Ricci principal direction of M^{n+1}, then either V^n is totally umbilical, or V^n is locally a subprojective Riemannian space $\eta^2S^{n-1}\times R^1$, where $\eta\in C(R^1)$; if V^n is conformally flat and all the principal normal curvatures are constants, then either V^n is totally umbilical, or V^n is locally a Riemannian product $S^{n-1}\times R^1$; and if the number of distinct principal normal curvatures of V^n is two and the multiplicities of them are l_1 and l_2 reepectively, and $l_1\geqslant 2$, $l_2\geqslant 2$, and the normel direction of V^n is a Ricci principal direction of M^{n+1}, then V^n is locally the Riemannian product $M_1^{l_1}\times M_2^{l_2}$ of two conformally flat Riemannian manifolds, moreover, when $l_1\geqslant 3$ (resp. $l_2\geqslant 3$), $M_1^{l_1}$ (resp. $M_2^{l_2}$) is of constant curvature. Because all the directions of S^{n+1} are Ricci principal directions, our results are generalizations of the results mentioned aboove.

§2. Let M^{n+1} be a conformally flat Riemannian manifold and M^n a conformally flat hypersurface with constant mean curvature in M^{n+1}. We assume that

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the normal direction of M^n is a Ricci principal direction of M^{n+1}. Let $e_1,\cdots,e_{n+1}$ be a local orthonormal frald field in M^{n+1} such that $e_1,\cdots,e_n$ are in the principal directions of M^n and e_{n+1} is normal to M^n. Let $\omega_1,\cdots,\omega_{n+1}$ be the dual frame field. We shall make use of the following convention on the ranges of indices:

$a, b, c, d=1,\cdots, n+1;\ i, j, k, l=1,\cdots, n;\ p, q, r, s=1,\cdots, n-1.$

Then we have the following stucture equations:

$$d\omega_a=-\sum_b \omega_{ab}\wedge\omega_b, \tag{1}$$

$$d\omega_{ab}=-\sum_c \omega_{ac}\wedge\omega_{cb}+\frac{1}{2}\sum_{c,d} K_{abcd}\omega_c\wedge\omega_d, \tag{2}$$

where

$$K_{abcd}=\frac{1}{n-1}(\delta_{ac}K_{bd}-\delta_{ad}K_{bc}+\delta_{bd}K_{ac}-\delta_{bc}K_{ad})+\frac{K}{n(n-1}(\delta_{ad}\delta_{bc}-\delta_{ac}\delta_{bd}) \tag{3}$$

is the curvature tensor of M^{n+1}, and

$$K_{ab}=\sum_c K_{abcd},\quad K=\sum_a K_{aa}$$

are Ricci tensor and scalar curvature of M^{n+1}, respectively. Restricting in M^n we have

$$\omega_{n+1}=0,\quad \omega_{n+1\,i}=\lambda_i\omega_i \tag{4}$$

$$d\omega_{ij}=-\sum_l \omega_{ik}\wedge\omega_{kj}+\frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k\wedge\omega_l, \tag{5}$$

$$R_{ijkl}=\lambda_i\lambda_j(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})+K_{ijkl}, \tag{6}$$

where R_{ijkl} is the curvature tensor of M^n, and λ_i the principal normal curvature in the direction of e_i. Let h_{ij} be the components of the second fundamental form. Then we have

$$h_{ij}=\lambda_i\delta_{ij} \tag{7}$$

We define the functions h_{ijk} in M^n such that

$$\sum_K h_{ijk}\omega_k=dh_{ij}-\sum_k h_{kj}\omega_{ki}-\sum_k h_{ik}\omega_{kj} \tag{8}$$

Because the noramal direction of M^n is a Ricci principal direction of M^{n+1}, we have $K_{n+1\,i}=K_{i\,n+1}=0$. Following [4] we have

$$h_{ijk}-h_{ikj}=-K_{n+1\,ijk}=0 \tag{9}$$

Pai Chenguo [5] proved that a hypersurface M^n $(n\geqslant 4)$ of conformally flat Riemannian manifold M^{n+1} is conformally flat if and onfy if there is a principal norml curvature with multiplicity not less than n-1 in each point of M^n. Hence, we have only the following two cases:

Case 1. $\lambda=\cdots=\lambda_n$, then M^n is totally umbilical.

Case 2. $\lambda_1=\cdots=\lambda_{n-1}=\mu\neq\lambda_n=\nu$. Then by means of (7) and (8) we have

$$\sum_k h_{pqk}\omega_k=0\ (p\neq q), \tag{10}$$

and, hence $h_{pqk}=0$, $(p\neq q)$. From (7)—(10) we obtain

$$d\mu=dh_{pp}=\sum_k h_{ppk}\omega_k=p_{ppp}\omega_p+h_{ppn}\omega_n \text{ and, therefore,}$$

$$h_{ppp}=0,\quad h_{ppn}=h_{qqn},\quad d\mu=h_{ppn}\omega \tag{11}$$

By means of (7) and (8), we have

$$d\nu=dh_{nn}=\sum_k h_{nnk}\omega_k.$$

By the assumption, the mean curvature $h=\frac{n-1}{n}\mu+\frac{\nu}{n}=$const., we have

$$\sum_k h_{nnk}\omega_k+(n-1)h_{ppn}\omega_n=0.$$

Hence,

$$h_{nnp}=0,\quad h_{ppn}=-\frac{1}{n-1}h_{nnn} \tag{12}$$

and
$$d\nu=h_{nnn}\omega_n \tag{13}$$

Thus, from (8) we have $h_{pnp}\omega_p=-\nu\omega_{np}-\mu\omega_{pn}=(\nu-\mu)\omega_{pn}$, and, therefore,

$$\omega_{pn}=\frac{1}{\nu-\mu}h_{ppn}\omega_p=\frac{h_{nnn}}{(n-1)(\mu-\nu)}\omega_p. \tag{14}$$

From (13), we obtain $dh_{nnn}\wedge\omega_n-h_{nnn}\sum_p\omega_{np}\wedge\omega_p=0$, and, hence,

$$dh_{nnn}=\alpha\omega_n \tag{15}$$

By means of (14), we have

$$d\omega_{pn}=\frac{1}{(n-1)(\mu-\nu)}\left\{\left[\alpha+\frac{h_{nnn}^2}{(n-1)(\mu-\nu)}\right]\omega_n\wedge\omega_p-h_{nnn}\sum_q\omega_{pq}\wedge\omega_q\right\} \tag{16}$$

On the other hand, from (2), (3) and (14), we have

$$d\omega_{pn}=-\sum_q\frac{h_{nnn}}{(n-1)(\mu-\nu)}\omega_{pq}\wedge\omega_q+\left[\mu\nu+\frac{1}{n-1}\left(K_{pp}+K_{nn}-\frac{K}{n}\right)\right]\omega_p\wedge\omega_n$$

$$+\frac{1}{n-1}\sum_q K_{nq}\omega_p\wedge\omega_q+\frac{1}{n-1}\sum_{p\neq q}K_{pq}\omega_q\wedge\omega_n. \tag{17}$$

Comparing (16) and (17), we get

$$K_{pq}=0\ (p\neq q),\quad K_{nq}=0,\quad (q=1,\cdots,n-1),$$

$$(n-1)\mu\nu+K_{pp}+K_{nn}-\frac{K}{n}=\frac{1}{(n-1)(\mu-\nu)^2}[(n-1)(\mu-\nu)\alpha$$

$$+h_{nn}^2],\quad (p=1,\cdots,n=1), \tag{18}$$

and, therefore,

$$K_{11}=K_{22}=\cdots=K_{n-1n-1}. \tag{19}$$

By means of (1) and (14), we have

$$d\omega_n=0.$$

Thus the Pfaffian equation $\omega_n=0$ is completely integrable. Let f be its first integral. Restricting in the integral manifold V^{n-1}: $f=c$, c is any constant, we have

$$\omega_{n+1}=0,\ \omega_n=0,\ \omega_{n+1p}=\mu\omega_p,\ \omega_{np}=\frac{h_{nnn}}{(n-1)(\nu-\mu)}\omega_p \tag{20}$$

Hence V^{n-1} is a totally umbilical hypersurface of M^n. Moreover, using (18) and (19), we have

$$d\omega_{pq}=-\sum_r\omega_{pr}\wedge\omega_{rq}+\Big\{\mu^2+\frac{h^2_{nnn}}{(n-1)^2(\mu-\nu)^2}+\frac{1}{n(n-1n^2}$$

$$\times[(n+1)K-2n(K_{nn}+K_{n+1\ n+1})]\Big\}\omega_p\wedge\omega_q$$

and, therefore, V^{n-1} $(n\geqslant4)$ is of constant sectional curvature. Thus, in the loca coordinates of M^n we have

$$ds^2=\sigma^2\sum_p(dx^p)^2+\tau^2(dx^n)^2, \tag{21}$$

where $xg=$const. is the integral manifolds of $\omega_n=0$. Now we have

$$\omega_p=\sigma dx^p,\ \omega_n=\tau dx^n, \tag{22}$$

and

$$d\omega_p=\sum_q\frac{1}{\sigma^2}\frac{\partial\sigma}{\partial x^q}\omega_q\wedge\omega_p+\frac{1}{\tau\sigma}\frac{\partial\sigma}{\partial x^n}\omega_n\wedge\omega_p. \tag{23}$$

By means of (11), (13) and (22), we have

$$\frac{\partial\mu}{\partial x^p}=0,\quad\frac{\partial\nu}{\partial x_p}=0,\quad\frac{\partial\mu}{\partial x^n}=\tau h_{ppn},$$

i. e.,

$$\mu=\mu(x^n),\ \nu=\nu(x^n),\ \mu'=\tau h_{ppn}. \tag{24}$$

Comparing (1) with (23) we obtain

$$\omega_{pq}=\frac{1}{\sigma^2}\frac{\partial\sigma}{\partial x^q}\omega_p,\quad\omega_{pn}=\frac{1}{\tau\sigma}\frac{\partial\sigma}{\partial x^n}\omega_p. \tag{25}$$

From (25), (14) and (24), we get

$$\frac{1}{\sigma}\frac{\partial\sigma}{\partial x^n}=\frac{\mu'}{\nu-\mu},$$

and, hence,

$$\sigma=\eta(x^n)\xi(x^1,\cdots,x^{n-1}). \tag{26}$$

By means of (1), (4), (14) and (22), we obtain

$$\tau=\tau(x^n). \tag{27}$$

Moreover, since all the hypersurfaces $x^n=$const. have constant sectional curvature, (21) is reducible to the following form:

$$ds^2=\eta^2(x^n)\frac{\sum_p(dx^p)^2}{\left[1+\frac{c}{4}\sum_p(x^p)^2\right]^2}+(dx^n)^2,$$

where c is constant, This shows that M^n is locally a subprojective Riemannian space $\eta^n s^{n-1}\times R^1$, $\eta\in C(R^1)$. Hence we have the following

Theorem 1. Let M^n $(n\geqslant 4)$ be a conformally flat hypersurface with constant mean curvature in a conformally flat Riemannian manifold M^{n+1}. Assume that the normal direction of M^n is a Ricci principal direction of M^{n+1}. Then either M^n is totally umbilical, or M^n is locally a subprojective Riemannian space $\eta^2 S^{n-1}\times R^1$, where $\eta\in C(R^1)$ and R^1 is a line.

If all the principal normal curvatures are constants, there is $\eta(x^n)=$const. in (28), and therefore, M^n is locally the Riemannian product $S^{n-1}\times R^1$. Moreover, by an argument analogous to get the theorem 1, we can get the following

Theorem 2. Let M^n $(\geqslant 4)$ be a conformally flat hypersurface with constant principal normal curvatures in a conformally flat Riemannian manifold M^{n+1}. Then either M^n is totally unbilical, or M^n is locally the Riemannian product $S^{n-1}\times R^1$.

When M^{n+1} is a Riemannian manifold of constant curuature the theorem 2 implies the result of [3] mentioned above.

§3. Let V^n be a hypersurface with constant mean curvature in a confomally falt Riemanniaa manifold M^{n+1}. Assume that the normal direction of distinct principal curvatures of V^n is two. If the multiplicity of one principal curvature is n-1, V^n is conformally flat [5]. We try to consider the following case;

$$\lambda_1=\cdots=\lambda_{l_1}=\mu_1,\ \lambda_{l_1+1}=\cdots=\lambda_{l_1+l_2}=\mu_2\neq\mu_1,\ l_1,\ l_2\geqslant 2,\ l_1+l_2=n. \tag{29}$$

By an argument analogous to get (10) and (11), we have now

$$h_{p_\alpha q_\alpha i}=0,\ h_{p_\alpha p_\alpha p_\alpha}=0,\ h_{p_\alpha p_\alpha p_\beta}=h_{q_\alpha q_\alpha q_\beta},$$
$$d\mu_\alpha=\sum_{p_\alpha}h_{p_\alpha p_\alpha p_\beta}\omega_{p_\beta},\quad\begin{pmatrix}\alpha,\beta=1,2;\ \alpha\neq\beta,\ p_\alpha\neq q_\alpha,\ p_\beta\neq q_\beta,\\ p_1,q_1=1,\cdots,l_1;\ p_2,q_2=l_1+1,\cdots,n\end{pmatrix} \tag{30}$$

From (9) and (30) we obtain

$$\mu_1=\text{const.},\ \mu_2=\text{const.}, \tag{31}$$

and, therefore,

$$h_{ijk}=0. \tag{32}$$

By means of (8), (9), (29) and (32), we obtain

$$\omega_{p_1p_2}=0 \tag{33}$$

Thus, the system of Pfaffian equations $\omega_{p_1}=0$ $(p_1=1,\cdots,l_1)$ is completely integrable. Let $f_1,\cdots,f_{l_1}$ be its independent first integrals. Similarly, the system of Pfaffian equations $\omega_{p_2}=0(p_2=l_1+1,\cdots,n)$ is completely integrable. Let $f_{l_1+1},\cdots f_n$ be itsindependent first integrals. It is easily seen that $f_1,\cdots,f_n$ are n independent functions and all the integral manifolds $V_1^{l_1}$: $f_{p_2}=$const. $(p_2=l_1+1,\cdots,n)$ and

V_2^{l}: $f_{p_1}=\text{const}$. $(p_1=1,\cdots,l_1)$ are totally geodesic in V^n and, therefore, are conformally flat (see [6]). Moreover we have

$$K_{ij}=0, \quad (i\neq j), \quad K_{p_\alpha p_\alpha}=K_\alpha, \quad \mu_1\mu_2=\frac{K}{n}-K_1-K_2. \tag{34}$$

Restricting in $V_\alpha^{l_\alpha}$ $(\alpha=1,2)$, we have

$$d\omega_{p_\alpha q_\alpha}=-\sum_{r_\alpha}\omega_{p_\alpha r_\alpha}\wedge\omega_{r_\alpha q_\alpha}+\frac{1}{n-1}\left\{2K_\alpha=\frac{K}{n}+(n-1)\mu_\alpha^2\right\}\omega_{p_\alpha}\wedge\omega_{q_\alpha}. \tag{35}$$

Hence, when $l_\alpha\geqslant 3$, $V_\alpha^{l_\alpha}$ is of constant curvature, Thus we have the following

Theorem 3. Let V^n $(n\geqslant 4)$ be a hypersurface with constant mean curvature in a conformally flat Riemannian manifold M^{n+1}. Assume that the normal direction of V^n is a Ricci principal direction of M^{n+1}. If the multiplicities of the principal normal curvatures of V^n are l_1 and l_2, $(l_1+l_2=n;\ l_1,l_2\geqslant 2)$, respectively, then V^n is locally the Riemannian product $M_1^{l_1}\times M_2^{l_2}$ of two confrmally flat manifolds. Moreover, when $l_1\geqslant 3$, (resp. $l_2\geqslant 3$), $M_1^{l_1}$ (resp. $M_2^{l_2}$) is of constant curvature.

If M^{n+1} is also locally symmetric, we have $K=\text{const.}$, and consequently by mean of(20)and (34) we see that if one of K_1 and K_2 is constant, so is another. Thus if $n\geqslant 5$, then one of l_1 and l_2 is not less than 3 and, therefore, both K_1 and K_2 are constants. Hence we have the following result which is a generalization of the result of [1] mentioned above.

Corollary. Let V^n $(n\geqslant 5)$ be a hypersurface with constant mean curvature in a locally symmetric and conformally flat Riemannian manifold M^{n+1}. Assume that the normal direction of V^n is a Ricci principal direction of M^{n+1}. If the multiplicities of the principal normal curvatures of V^n are l_1 and l_2, $(l_1+l_2=n;\ l_1,l_2\geqslant 2)$, respectively, then V^n is locally the Riemannian product $S_1^{l_1}\times S_2^{l_2}$ of two Riemannian manifold of constant curvature.

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