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A NUMERICAL METHOD FOR OPTIMAL INPUT DESIGN BY FREQUENCY DOMAIN CRITERIA

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(I) Introduction

Most of the existing literature on input design for identification has been concentrated on the problem of obtaining accurate parameter estimates. Many of them discussed how to determine optimal inputs which minimize a scalar valued function of the inverse Fisher information matrix. The minimization has been performed under certain constraints to prevent the diverge of the input or output amplitude.

Even for the frequency domain design problem, optimal inputs are also determined by minimizing a scalar valued function of the covariance matrix of parameter estimates[1], [2].

However, the aim of identification is usually to know the input-output properties of a system. If the accuracy of the parameter estimates affects the dynamic characteristics of the system, the previous optimal inputs are useful. In some situation where the parameters of a model do not have physical significance, the accuracy of the parameter estimates as such are without interest. For example, when we were doing some simulations of the adaptive control system, we found that the parameter estimates could be changed a quite lot but the dynamics of the system did not change much. The accuracy of the parameter estimates is then not so interesting.

If the aim of identification is to design a control system using a frequency domain method, the accuracy of the parameter estimates does not mean the accurate representation in frequency response or transfer function, because small errors in one representation may give large errors in another, the previous optimal inputs are not suitable.

In this paper we consider the optimal input design by a frequency domain criterion. This problem is formulated in section II. The new problem can be inverted into the well known problem. This calculation is given in section III. The section IV and V give the calculations of the criterion function and the solution of the minimization problem in detail. We give some numerical example in section VI and the conclusions in section VII.

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(II) Problem formulation

In this section we shall consider the following formulation. Let the model set be given by

$$y(t)=G(\theta,\ q^{-1})u(t)+H(\theta,\ q^{-1})e(t), \tag{1}$$

where G and H are rational functions in the delay operator q^{-1}. The disturbance sequence $\{e(t)\}$ is supposed to be white noise. The output is $y(t)$ and the input is $u(t)$. The true system is supposed to be given by

$$y(t)=G_0(q^{-1})u(t)+H_0(q^{-1})e(t). \tag{2}$$

If

$$G(\theta,\ q^{-1})=\frac{B(q^{-1})}{A(q^{-1})},\ H(\theta,\ q^{-1})=\frac{\lambda C(q^{-1})}{A(q^{-1})},$$

where

$$A(q^{-1})=1+a_1q^{-1}+\cdots+a_Mq^{-M},\ B(q^{-1})=b_1q^{-1}+\cdots+b_Mq^{-M},$$
$$C(q^{-1})=1+c_1q^{-1}+\cdots+c_Mq^{-M}.$$

Eq. (1) can be written as

$$A(q^{-1})y(t)=B(q^{-1})u(t)+\lambda C(q^{-1})e(t) \tag{3}$$

and

$$\theta^T=(a_1\cdots a_Mb_1\cdots b_Mc_1\cdots c_M).$$

This is the well known ARMAX model. The transfer function estimate of (1) can be given by $G(\hat{\theta}_N,\ e^{i\omega})$. The parameter estimate $\hat{\theta}_N$ is obtained using the prediction error identification method.

The optimal input desgin problem can now be formulated as

$$\min_{\phi_u(\omega)} V(\phi_u(\omega)), \tag{4a}$$

where

$$V(\phi_u(\omega))=E\int_{-\pi}^{\pi}|G(\hat{\theta}_N,\ e^{i\omega})-G_0(e^{i\omega})|^2Q(\omega)d\omega \tag{4b}$$

under the constraint

$$\int_{-\pi}^{\pi}\phi_u(\omega)d\omega\leqslant C_\pi \tag{5}$$

Here $\phi_u(\omega)$ is the power spectral density of the input signal. We thus confine ourselves to such inputs for which $\phi_u(\omega)$ is well defined. The function $Q(\omega)$ is a weighting function chosen by the user, and 'E' denotes the expectation with respect to the estimate $\hat{\theta}_N$. The constraint (5) means that the input has bounded variance.

(III) Calculations of the cost function

The criterion function (4b) can be converted into the well known criterion function of the input design problem. The criterion function is also called the cost function when we are solving the optimization problem.

If we assume that for some θ_0

$$G(\theta_0, q^{-1})=G_0(q^{-1}),\ H(\theta_0, q^{-1})=H_0(q^{-1}).$$

Due to

$$G(\hat{\theta}_N, e^{i\omega})-G_0(e^{i\omega})=\frac{\partial G}{\partial\theta}\bigg|_{\theta=\theta_0}(\hat{\theta}_N-\theta_0),\ \text{as } \hat{\theta}_N\to\theta_0,$$

$$E\int_{-\pi}^{\pi}|G(\hat{\theta}_N, e^{i\omega})-G_0(e^{i\omega})|^2Q(\omega)\mathrm{d}\omega=$$

$$=E\int_{-\pi}^{\pi}\frac{\partial G}{\partial\theta}\bigg|_{\theta=\theta_0}(\hat{\theta}_N-\theta_0)(\hat{\theta}_N-\theta_0)^T\left(\frac{\partial G}{\partial\theta}\bigg|_{\theta=\theta_0}\right)^*Q(\omega)\mathrm{d}\omega$$

$$=\text{trace}\left\{E(\hat{\theta}_N-\theta_0)(\hat{\theta}_N-\theta_0)^T\int_{-\pi}^{\pi}\left(\frac{\partial G}{\partial\theta}\bigg|_{\theta=\theta_0}\right)^*\frac{\partial G}{\partial\theta}\bigg|_{\theta=\theta_0}Q(\omega)\mathrm{d}\omega.\right\}$$

Therefore

$$\lim_{N\to\infty}E\int_{-\pi}^{\pi}|G(\hat{\theta}_N, e^{i\omega})-G_0(e^{i\omega})|^2Q(\omega)\mathrm{d}\omega=\text{trace}P\cdot W, \tag{6}$$

where

$$P=\lim_{N\to\infty}\text{Cov}(\hat{\theta}_N), \tag{7}$$

$$W=\int_{-\pi}^{\pi}\left(\frac{\partial G}{\partial\theta}\bigg|_{\theta=\theta_0}\right)^*\frac{\partial G}{\partial\theta}\bigg|_{\theta=\theta_0}Q(\omega)\mathrm{d}\omega \tag{8}$$

It is easy to see that W is a Hermitian matrix, i. e. $W^*=W$, where '*' denotes conjugate and transpose.

This trace has following properties,

1. trace $PW\geqslant 0$. In view of (9)

$$\text{trace } PW=\int_{-\pi}^{\pi}\frac{\partial G}{\partial\theta}\bigg|_{\theta=\theta_0}\text{Cov}(\hat{\theta}_N)\left(\frac{\partial G}{\partial\theta}\bigg|_{\theta=\theta_0}\right)^*Q(\omega)\mathrm{d}\omega,$$

$\text{Cov}(\hat{\theta}_N)$ is a non-negative definite matrix and $Q(\omega)$ a non-negative valued function. The integrated function is non-negative, so the trace is non-negative as well.

2. The Hermitian matrix W can be written as $W=W_r+iW_i$, where W_r anp W_i are real matrices, 'i' is an unit imaginary number.

It is easy to verify trace $P\cdot W_i=0$. Therefore

$$\text{trace } PW=\text{trace } P\cdot W_r. \tag{10}$$

(IV) The parametrization of the input signal

Let the set D be defined by

$$D=\left\{u(t)\bigg|\int_{-\pi}^{\pi}\phi_u(\omega)\mathrm{d}\omega=C_u\right\},$$

where C_u is a constant. For simplicity, let $C_u=1$. The parametrization of the input signal considered in [2]-[4] is of the form

$$u_1(t)=\sum_{k=1}^{2n}\alpha_k\sin\omega_k t.$$

It has been proven that for any $u(t)\in D$ there exists an $u_1(t)\in D$ obtained by suitablely choosing $\{\alpha_k\}$ and $\{\omega_k\}$ such that $V(\phi_u(\omega))=V(\phi_{u_1}(\omega))$, see [4]. So we take $u(t)=u_1(t)$.

Now, we consider the normalized input signal

$$u(t)=\sum_{k=1}^{2n}\alpha_k\ \sin\omega_k t\ \sqrt{\frac{2}{\sum_{k=1}^{2n}\alpha_k^2}} \tag{11}$$

The power spectral density of $u(t)$ is

$$\phi_u(\omega)=\frac{1}{2}\sum_{k=1}^{2n}\frac{\alpha_k^2}{\sum_{k=1}^{2n}\alpha_k^2}\{\delta(\omega-\omega_k)+\delta(\omega+\omega_k)\}, \tag{12}$$

where $0\leqslant\omega_k\leqslant\pi$ and

$$\int_{-\pi}^{\pi}\phi_u(\omega)d\omega=1. \tag{13}$$

For the ARMAX model, the dimension of θ is $3M$. Let

$$\varphi(t)^T=(-y(t-1)\cdots-y(t-M)u(t-1)\cdots u(t-M)). \tag{14}$$

The matrix R is defined by

$$R=\frac{1}{N}\sum_{t=M+1}^{M+n}\varphi(t)\varphi^T(t). \tag{15}$$

According to Ljung and Caines (1979), we obtain

$$P=(E\varphi(t)\varphi^T(t))^{-1}. \tag{16}$$

Suppose that

$$\lim_{N\to\infty}R=E\varphi(t)\varphi^T(t). \tag{17}$$

Then

$$P=R^{-1},\ \text{as}\ N\to\infty. \tag{18}$$

From (11), the correlation funcsion of $u(t)$ can be obtained

$$r_u(\tau)=\sum_{k=1}^{2n}\frac{\alpha_k^2}{\sum_{l=1}^{2n}\alpha_l^2}\cos(\tau\omega_k),\quad \tau=0,\ 1,\cdots,\ M-1. \tag{19}$$

Calculate

$$\phi_{uy}(\omega)=\phi_u(\omega)\ G_0(e^{i\omega}),$$

$$r_{uy}(\tau)=\int_{-\pi}^{\pi}\phi_u(\omega)G_0(e^{i\omega})e^{i\tau\omega}d\omega, \tag{20}$$

where $\phi_u(\omega)$ is given by (12) and $G_0(e^{i\omega})=B_0(e^{-i\omega})/A_0(e^{-i\omega})$.

Solving the following equations (21), we obtain $r_y(\tau)$, $\tau=0,\cdots,M-1$.

$$\begin{cases} r_y(0)+a_1r_y(1)+\cdots+a_Mr_y(M)=b_1r_{yu}(1)+b_2r_{yu}(2)+\cdots+b_Mr_{yu}(M)+r_{00}, \\ r_y(1)+a_1r_y(0)+\cdots+a_Mr_y(M-1)=b_1r_{yu}(-1)+b_2r_{yu}(0)+\cdots+b_Mr_{yu}(M-1)+r_{01}, \\ \vdots \\ r_y(M)+a_1r_y(M-1)+\cdots+a_Mr_y(0)=b_1r_{yu}(M-1)+b_2r_{yu}(M-2)+\cdots+b_Mr_{yu}(0) \\ \qquad +r_{0M}, \end{cases} \tag{21}$$

where

$$\begin{cases} r_{00}=Ey(t)\cdot\lambda\cdot[e(t)+c_1e(t-1)+\cdots+c_Me(t-M)], \\ r_{01}=Ey(t-1)\lambda[e(t)+c_1e(t-1)+\cdots+c_Me(t-M)], \\ \vdots \\ r^{0M}=Ey(t-M)\lambda[e(t)+c_1e(t-1)+\cdots+c_Me(t-M)]=\lambda^2c_M \end{cases}$$

and

$$r_{yu}(-k)=r_{uy}(k).$$

Then the correlation matrix is obtained

$$R=\begin{bmatrix} r_y(0) & r_y(1) & \cdots & r_y(M-1) & -r_{yu}(0) & -r_{yu}(1) & \cdots & -r_{yu}(M-1) \\ \vdots & & & & & & & \\ r_y(M-1) & r_y(M-2) & \cdots & r_y(0) & -r_{uy}(M-1) & -r_{uy}(M-2) & \cdots & -r_{yu}(0) \\ -r_{yu}(0) & -r_{uy}(1) & \cdots & -r_{uy}(M-1) & r_u(0) & r_u(1) & \cdots & r_u(M-1) \\ \vdots & & & & & & & \\ -r_{yu}(M-1) & -r_{yu}(M-2) & \cdots & -r_{yu}(0) & r_u(M-1) & r_u(M-2) & \cdots & r_u(0) \end{bmatrix} \tag{22}$$

From (18), we obtain P.

(V) Optimization

From (6), (10) and (12), we obtain

$$V(\phi_u(\omega))=\text{trace } PW_r=V(\eta) \tag{24}$$

where

$$\eta^T=(\alpha_1\cdots\alpha_M,\ \omega_1\cdots\omega_M)$$

The optimal input design problem (4), (5) has been converted into an unconstrainted optimization problem

$$\min_{\eta\in R^{2m}} V(\eta) \tag{25}$$

E04CGF is an easy-to-use quasi-Newton algorithm for finding an unconstrainied minimum of $V(\eta)$ of 2M independent variables $\alpha_1,\cdots,\alpha_M$, $\omega_1,\cdots,\omega_M$ using function-values only.

(VI) Numerical examples

Example. Consider a 2-order time invariant linear system

$$y(t)-1.5y(t-1)+0.7y(t-2)=u(t-1)+0.5u(t-2)+\lambda e(t)$$

where $\{e(t)\}$ is a sequence of zero mean rectangle distributed and independent random variables. Using the above programe, we obtain the coefficients of the optimal input parmetrization (table)

λ	α_k	ω_k	criterion function
1	2.87408 2.87088 0.01312 0.02361	−0.00740 −0.01238 1.47512 1.42306	0.03704
$\sqrt{12}$	10.56161 4.76739 −0.20692 −0.15994	0.01779 0.04017 1.00663 0.53629	0.44454

The optimal input is given by

$$u(t)=\sum_{k=1}^{4}\alpha_k\sin(\omega_k t).$$

(VII) Conclusions

The frequency criterion function can be calculated from the Hermitain matrix W and the covariance matrix P when the number of data, N, tends to infinity. The optimal input design problem is inverted into an unconstrainted optimization problem and the optimal input signal is the linear combination of some sine waves.

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