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Numbers of Functional Lattices*

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Definition 1 Let L be a complete lattice, X be an infinite set, $L^X=\{f: f$ is a function & domain $f=X$ & range $f\subseteq L\}$, Define operations of lattice $\vee,\wedge$; $(f\vee g)(x)=f(x)\vee g(x)$, $(f\wedge g)(x)=f(x)\wedge g(x)$, $(\vee\{f_i:i\in I\})(x)=\vee\{f_i(x):i\in I\}$, $(\wedge\{f_i:i\in I\})(x)=\wedge\{f_i(x):i\in I\}$.

Definition 2 $\tau(\subseteq L^X)$ is a functional lattice (or a $\vee$-closed and $\wedge$-finite closed lattice) iff $(\forall\Sigma\,\forall a\,(a\in\Sigma\,\&\,f_a\in\tau)\rightarrow\vee\{f_a, a\in\Sigma\}\in\tau)\,\&\,(\forall f,g\in\tau)(f\wedge g\in\tau)\,\&\,(((\forall x\in X)(f_0(x)=0))\rightarrow(f_0\in\tau))\,\&\,(((\forall x\in X)(f_1(x)=1))\rightarrow(f_1\in\tau))$. $\mathbf{C}=\{\tau:\tau\subseteq L^X\,\&\,\tau$ is a functional lattice$\}$. $\tau(\subseteq L^X)$ is a complete lattice of functions iff (τ is a functional lattice & $((\forall\Sigma\,\forall a\,(a\in\Sigma\,\&\,f_a\in\tau))\rightarrow(\wedge\{f_a:a\in\Sigma\}\in\tau)))$, $\mathbf{K}=\{\tau:\tau\subseteq L^X\,\&\,\tau$ is a complete lattice of functions$\}$.

Definition 3 For $\tau,\sigma\in\mathbf{C}$ define $\tau\cong\sigma$ iff $\exists\varphi(\varphi$ is a lattice-isomorphism from τ to $\sigma)$, $\hat{\tau}=\{\sigma:\sigma\in\mathbf{C}\,\&\,\sigma\cong\tau\}$; $\tau\sim\sigma$ iff $\exists\varphi(\varphi$ is a bijection on X & φ is continuous & φ^{-1} is continuous), where φ is continuous iff $((\forall g\in\sigma)(\varphi^{-1}(g)),\ (\forall x\in X)\ ((\varphi^{-1}(g))(x)=g(\varphi(x)))$, φ^{-1} is continuous iff $((\forall f\in\tau)(\varphi(f)\in\sigma)),\ (\forall x\in X)$

$$((\varphi(f))(x)=\begin{cases}\vee\{f(z):z\in\varphi^{-1}[\{y\}]\}, & \text{where } \varphi^{-1}[\{y\}]\neq\phi\ (\text{empty})\\ 0, & \text{else}\end{cases}),$$

$\langle\tau\rangle=\{\sigma:\sigma\in\mathbf{C}\,\&\,\sigma\sim\tau\}$. $\mathbf{C}_1=\{\hat{\tau}:\tau\in\mathbf{C}\}$, $\mathbf{C}_2=\{\langle\tau\rangle:\tau\in\mathbf{C}\}$, $\mathbf{K}_1=\{\hat{\tau}:\tau\in\mathbf{K}\}$, $\mathbf{K}_2=\{\langle\tau\rangle:\tau\in\mathbf{K}\}$. $[0,1]=\{x:x$ is a real number & $0\leqslant x\leqslant 1\}$.

Theorem 1—3 If $2\leqslant|L|\leqslant 2^{|X|}$, then $|\mathbf{C}|=|\mathbf{C}_1|=\mathbf{C}_2|\exp(\exp(|X|))=2^{|X|}$

Theorem 4—6 If $2\leqslant|L|\leqslant|X|$ and $2(\exists Q\subset L)(\forall l\in L)(\exists H\subset Q)(\vee H=\sup H=l)$ then $|\mathbf{K}|=|\mathbf{K}_1|=|\mathbf{K}_2|=\exp(|X|)=2^{|X|}$

Theorem 7—9 If $L=[0,1]$, then $|\mathbf{C}|=|\mathbf{C}_1|=|\mathbf{C}_2|=\exp(\exp(|X|))=2^{2^{|X|}}$

Theorem 10—12 If $L=[0,1]$, then $|\mathbf{K}|=|\mathbf{K}_1|=|\mathbf{K}_2|=\exp(|X|)=2^{|X|}$

References

[1] Birkhoff, G., Lattice Theory, 1948 (revised edition), p.49—64.

[2] Engelking, R., General Topology, 1977, p.16., p.27.

[3] Yang An-Zhou, Numbers of Complete sublattice of sets in the power set of an infinite set, J. Math. Res. & Exposition, 1986, №. 1, p.84.

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