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A Note on First Order Differential Equation*

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In this note, a theorem and its three corollaries on solution of the first order ordinary differential equation are given.

Theorem Suppose that b, $F\in C$, $a\in C^1$, $b(y)\neq 0$. If $a(t)$ and $b(t)$ satisfy the equality

$$a'(t)b(t)=1, \tag{1}$$

then the first order differential equation

$$y'=b(y)F(x,a(y)) \tag{2}$$

has a solution

$$y=f(u) \tag{3}$$

where $u=u(x)$ is a solution of the equatien

$$u'=F(x,u) \tag{4}$$

and $f(u)$ is any solution of the equation

$$f'(u)-b(f(u))=0. \tag{5}$$

Proof. Suppose that $u=u(x)$ is a sdution of (4). From (3), we obtain

$$y'=\frac{df}{du}u'. \tag{6}$$

Putting

$$\frac{df}{du}=b(y),\quad u=\int\frac{dy}{b(y)} \tag{7}$$

and from (6), (7) and (1) we get

$$u'(x)=\frac{y'}{b(y)},\quad a(t)=\int\frac{dt}{b(t)},\quad u=a(y). \tag{8}$$

Substituting (8) into (4) we obtain the equation (2). Using (3) and the left hand side of (7) we obtain the equation (5).

Remark I By $u=a(y)$ the equation (2) becomes (4). In this case, for instance, from $u=\ln(4y^3+6y^2+3y+\frac{1}{2})$ it is not easy to find $y=h(u)$. On

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the other hand, it is easy to find $f(u)=C\exp\frac{1}{3}u-\frac{1}{2}$ if $b(y)=\frac{1}{3}y+\frac{1}{6}$.

Remark 2 For the case $F(x,u)=-pu-q$ we obtain a result in [1].

Corollary 1 If the general solution of (4) is $u=\varphi(x,C)$, then $y=f(\varphi(x,C))$ is a bunch of solution of (2).

Corollary 2 If the equation (4) has a periodic solution, then (2) has periodic solutions too.

Example 1 The equation

$$y'=y[\ln^2 y+2(\sin x-1)\ln y+\sin^2 x-2\sin x-\cos x+1] \tag{9}$$

is of the same type as (2), here $b(y)=y$, $a(y)=\ln y$, and $a'(y)b(y)=1$, i.e.,the condition (1) is satisfied. Since the equation of Riccati type

$$u'=u^2+2(\sin x-1)u+\sin^2 x-2\sin x-\cos x+1 \tag{10}$$

is of the same type as (2.7) in [2], whose general solution is

$$u=1-\sin x-\frac{1}{x+C}, \tag{11}$$

and the equation (5) is of the form

$$f'(u)-f(u)=0$$

which has a solution $f(u)=e^u$, by using Corollary 1 the function

$$y=\exp(1-\sin x-\frac{1}{x+C}) \tag{12}$$

is a bunch of solution of (9).

In addition, it is easy to see that the equation (10) has a unique periodic solution[3] $u=1-\sin x$, hence, by using Corollary 2 the equation (9) has a periodic solution $y=\exp(1-\sin x)$.

Corollary 3 Suppose that a solution $u=u(x)$ of (4) satisfies the inequalities

$$g(x)<u(x)<h(x) \tag{13}$$

on (x_0,∞) and that a solution $f=f(u)$ of (5) is a strictly increasing function, then the solution $y=y(x)$ of (2) satisfies the inequalities

$$f(g(x))<y(x)<f(h(x)). \tag{14}$$

This corollary shows that the estimation of the solution of (2) can be obtained by that of (4).

Example 2 The solution $y=y(x)$ of the initial value problem

$$y'=xy-y\ln^2 y,\quad y(4)=e^2 \tag{15}$$

satisfies the inequalities

$$\exp(\sqrt{x}-0.07)<y(x)<\exp\sqrt{x}\qquad (x>4)$$

and the solution $y=y(x)$ of the initial value problem

$$y'=xe^{-y}-e^y,\quad y(4)=\ln 2 \tag{16}$$

satisfies the inequalities

$$\ln(\sqrt{x}-0.07)<y(y)<\ln\sqrt{x}\qquad (x>4)$$

because the solution $u=u(x)$ of the initial value problem

$$u'=x-u^2, \quad u(4)=2$$

satisfies for $x>4$ the inequalities[4]

$$\sqrt{x}-0.07<u(x)<\sqrt{x}.$$

Here a solution $f=f(u)$ of (5) is of the form e^u, $\ln u$ respectively.

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