一类正线性算子的渐近常数*

周信龙
(杭州大学)

§1 主要结果

设C[A, B]上的正线性算子

$$L_n(f;\ x)=\int_A^B f(u)\,\mathrm{d}\sigma_{n,x}(u),\quad \forall f\in C[A,\ B].$$

满足$\int_A^B \mathrm{d}\sigma_{n,x}(u)=1$，并且$\lim\limits_{n\to\infty}L_n(f;\ x)=f(x)$．记

$$T_{n_j}(x)=\int_A^B (u-x)^j\mathrm{d}\sigma_{n,x}(u),\quad W(f;\ \delta)=\sup_{\substack{|x-y|\leqslant\delta\\ x,y\in[A,B]}}|f(x)-f(y)|.$$

$$\mathrm{lip}_M\alpha=\{f\in C[A,\ B]:\ w(f;\ \delta)\leqslant M\delta^\alpha\},\quad \mathrm{Lip}_1\alpha\equiv\mathrm{Lip}\alpha,$$

$$E(\alpha,\ M,\ L_n,\ x)=\sup_{f\in\mathrm{lip}_M\alpha}|L_n(f;\ x)-f(x)|. \tag{1.1}$$

$$c(\psi,\ L_n,\ x)=\inf\left\{c:\ |L_n(f;\ x)-f(x)|\leqslant cw\left(f;\ \sqrt{\frac{\psi(x)}{n}}\right)\right\}, \tag{1.2}$$

其中$\psi(x)$是定义在[A, B]上的非负连续函数． $T_{n2}(x)=\dfrac{\varphi_n(x)}{n}$, $\lim\limits_{n\to\infty}\varphi_n(x)=\varphi(x)$，则记$c\varphi$, L_n, $x)=c_n(x)$．

不少作者对特殊的L_n研究了(1.1)(1.2)的渐近性质[1]－[4]．例如Essen[1]对Bernstein多项式算子B_n证得

定理 E

$$\overline{\lim_{n\to\infty}}\max_{0<x<1}\frac{|B_n(f;\ x)-f(x)|}{w\left(f;\ \frac{1}{\sqrt{n}}\right)}\leqslant B,\ B\text{的最佳值是}$$

$$B=2\sum_{j=0}^{\infty}(j+1)\{\Phi(2j+2)-\Phi(2j)\}=1.045564\cdots,$$

其中$\Phi(x)=\dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{t^2}{2}}\mathrm{d}t$.

Rathore和Singh[3]对参数为r的Gamma—型算子$G_n^r(f;\ x)$证得

定理 R-S $\lim\limits_{n\to\infty}n^{\frac{\alpha}{2}}\sup\limits_{f\in\mathrm{Lip}_M\alpha}|G_n^r(f;\ x)-f(x)|=M\dfrac{\Gamma\left(\frac{1+\alpha}{2}\right)}{\sqrt{\pi}}(2x^2)^{\frac{\alpha}{2}}$.

$$\lim_{n\to\infty}c(\psi,\ G_n^r,\ x)=2\sum_{j=0}^{\infty}(j+1)\left\{\Phi\left(\frac{\sqrt{\psi(x)}}{x}(j+1)\right)-\Phi\left(\frac{\sqrt{\psi(x)}}{x}j\right)\right\},$$

其中$\Phi(x)$的定义如前．下同．

最近，我们[5]对指数型算子$M_n(f;\ x)$得到了类似的结果．容易明白Berenstein多项式算子和当$r=-1$时的Gamma—型算子[6]是指数型算子．本文的目的是就更广的一类正线性

* 1984年1月10日收到.

算子讨论类似的问题. 所得定理包含了[1]－[4]中的相应结果. 我们有

定理 I 若

$$\lim_{n\to\infty}\frac{T_{n,2j}(x)}{(T_{n,2}(x))^{j}}=(2j-1)!!\ ,\qquad (j=1,2,\cdots)\ . \tag{1.3}$$

$$\lim_{n\to\infty}\frac{T_{n,2j+1}(x)}{(T_{n,2}(x))^{j+\frac{1}{2}}}=0\ ,\qquad (j=0,1,\cdots) \tag{1.4}$$

则

$$\lim_{n\to\infty} T_{n,2}^{-\frac{a}{2}}(x)\ E(a,M,L_n,x)=M\frac{\Gamma(\frac{1+a}{2})}{\sqrt{\pi}}2^{\frac{a}{2}}\ . \tag{1.5}$$

若$T_{n,2}(x)=\frac{\varphi_n(x)}{n}$, $\lim\limits_{n\to\infty}\varphi_n(x)=\varphi(x)$, 则当x不是ψ, φ的零点时

$$\lim_{n\to\infty}c(\psi,\ L_n,\ x)=2\sum_{j=0}^{\infty}(j+1)\left\{\Phi\left(\left(\frac{\psi}{\varphi}\right)^{\frac{1}{2}}(j+1)\right)-\Phi\left(\left(\frac{\psi}{\varphi}\right)^{1/2}j\right)\right\}\ . \tag{1.6}$$

证明 由于$L_n(f;\ x)$是正线性算子, 因此, 对每一固定的x, n, $\sigma_{n,x}(u)$是u的非负单调增加函数. 而且可以认为$\sigma_{n,x}(A)=0$, $\sigma_{n,x}(B)=1$. 定义$\sigma_{n,x}^{*}(u)$

$$\sigma_{n,x}^{*}(u)=\begin{cases}\sigma_{n,x}(u)\ ,\ u\in[A,B];\\ 0\ ,\ u<A;\\ 1,\ u>B.\end{cases}$$

我们看到$L_n(f,\ x)=\int_{-\infty}^{+\infty}f(u)\ d\sigma_{n,x}^{*}(u)$.

对每一实数Z, 让我们考虑函数

$$f_n(z)=\int_A^B e^{iz(u-x)T_{n,2}^{-1/2}(x)}d\sigma_{n,x}(u)=\int_{-\infty}^{+\infty}e^{iz(u-x)T_{n,2}^{-1/2}(x)}d\sigma_{n,x}^{*}(u)\ . \tag{1.7}$$

我们有$\lim\limits_{n\to\infty}f_n(z)=e^{-z^2/2}$. 事实上, 由 (1.3) 及Taylor公式

$f_n(z)=\sum\limits_{j=0}^{2N-1}\frac{(iz)^j}{j!}f_n^{(j)}(0)+\frac{(iz)^{2N}}{(2N)!}f_n^{(2N)}(\xi)$, $|f_n^{2N}(\xi)|\leqslant\int_A^B(u-x)^{2N}T_{n,2}^{-N}(x)\ d\sigma_{n,x}(u)=\frac{T_{n,2N}(x)}{T_{n,2}^{N}(x)}$

从而对任意$\varepsilon>0$, 存在N_0当 $n\geq N_0$时

$$\left|f_n(z)-\sum_{j=0}^{2N-1}\frac{(iz)^j}{j!}f_n^{(j)}(0)\right|\leq 2\frac{(z^2/2)^N}{N!}$$

(1.4) 表明当j是奇数时$f_n^{(j)}(0)\to 0$ $(0\to\infty)$. 若$j(=2\nu)$是偶数, 那么由 (1.3) 知 $\lim\limits_{n\to\infty}f_n^{(j)}(0)=\lim\limits_{n\to\infty}f_n^{(2\nu)}(0)=(-1)^{\nu}(2\nu-1)!!$.因此

$$\lim_{n\to\infty}f_n(z)=\sum_{j=0}^{\infty}\frac{(-z^2/2)^j}{j!}=e^{-z^2/2}\ .$$

注意到 (1.7) 立即有

$$\lim_{n\to\infty}\sigma_{n,x}^{*}(x+T_{n,2}^{1/2}(x)\ \ y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y}e^{-t^2/2}dt.$$

从而对任意有界连续函数$f(x)$

$$\lim_{n\to\infty}\int_{-\infty}^{+\infty}f((u-x)\ T_{n,2}^{-1/2}(x))\,d\sigma_{n,x}^{*}(u)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f(y)\ e^{-y^2/2}dy \tag{1.8}$$

不仅如此, 我们还有

$$\lim_{n\to\infty}\int_{-\infty}^{+\infty}|(u-x)\ T_{n,2}^{-1/2}(x)|^{\alpha}d\sigma_{n,x}^{*}(u)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}|y|^{\alpha}e^{-y^2/2}dy. \tag{1.9}$$

事实上，

$\int_{-\infty}^{+\infty}|(u-x)\ T_{n,2}^{-1/2}(x)|^{\alpha}d\sigma_{n,x}^{*}(u)=\int_{-\infty}^{+\infty}|y|^{\alpha}d\sigma_{n,x}^{*}(x+T_{n,2}^{1/2}(x)y)=\int_{-N}^{N}|y|^{\alpha}d\sigma_{n,x}^{*}(x+T_{n,2}^{1/2}(x)y)+$
$\int_{-\infty}^{-N}|y|^{\alpha}d\sigma_{n,x}^{*}(x+T_{n,2}^{1/2}(x)\ y)+\int_{N}^{\infty}|y|^{\alpha}d\sigma_{n,x}^{*}(x+T_{n,2}^{1/2}(x)\ y)\equiv I_1+I_2+I_3$;

$$I_3\leq\frac{1}{N^{2-\alpha}}\int_{-\infty}^{+\infty}y^2d\sigma_{n,x}^{*}(x+T_{n,x}^{1/2}(x)\ y)=\frac{1}{N^{2-\alpha}},$$

类似地$I_2\leq\frac{1}{N^{2-\alpha}}$．因此，（1.9）可以（1.8）推出．现在转向定理1的证明．显然

$$\sup_{f\in \mathrm{Lip}_M\alpha}|L_n(f;\ x)-f(x)|=M\int_A^B|u-x|^{\alpha}d\sigma_{n,x}(u)\ .$$

故

$$E(\alpha,\ M,\ L_n,\ x)=MT_{n,2}^{\alpha/2}(x)\int_A^B\left|\frac{(u-x)}{T_{n,2}^{1/2}(x)}\right|^{\alpha}d\sigma_{n,x}(u)\ .$$

于是（1.5）可以从（1.9）得到．为证（1.6），我们需要如下的结论[3]：

$$c(\psi,\ L_n,\ x)=1+\int_{-\infty}^{+\infty}[|u-x|(\frac{n}{\psi})^{1/2}]d\sigma_{n,x}^{*}(u)\ .$$

因此

$$c(\psi,L_n,x)=1+\int_{-\infty}^{+\infty}[|y|(\frac{\phi_n}{\psi})^{1/2}]d\sigma_{n,x}^{*}(x+T_{n,2}^{1/2}(x)\ y)\ .$$

记$\Omega_n(y)=\sigma_{n,x}^{*}(x+T_{n,2}^{1/2}(x)\ y)$，$\Omega(y)=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{y}e^{-\frac{t^2}{2}}dt$．我们要证明当$x$不是$\varphi$，$\psi$的零点时，

$$\lim_{n\to\infty}\int_{-\infty}^{+\infty}[|y|(\frac{\varphi_n}{\psi})^{1/2}]\ d\Omega_n(y)=\int_{-\infty}^{+\infty}[|y|(\frac{\varphi}{\psi})^{1/2}]\ d\Omega(y)\ . \tag{1.10}$$

为此，设$f_n(x)=(\frac{\psi_n(x)}{\psi(x)})^{1/2}$，$f(x)=(\frac{\varphi(x)}{\psi(x)})^{1/2}$．显然，当$n>n_0$时$f_n(x)$恒正．但

$$\int_0^N[|y|f_n]\ d\Omega_n(y)=\sum_{j=0}^{[Nf_n]}j\int_{\frac{j}{f_n}}^{\frac{j+1}{f_n}}d\Omega_n(y)-\int_N^{\frac{[Nf_n]+1}{f_n}}[|y|f_n]\ d\Omega_n(y)\ , \tag{1.11}$$

$$\int_0^N[|y|f]\ d\Omega_n(y)=\sum_{j=0}^{[Nf]}j\int_{\frac{j}{f}}^{\frac{j+1}{f}}d\Omega_n(y)-\int_N^{\frac{[Nf]+1}{f}}[|y|f]\ d\Omega_n(y)\ . \tag{1.12}$$

容易看出（1.11)(1.12）右边第二项均是$O(\frac{1}{N})$．另外，对固定的N只要$n>n_1$，$\max\{[Nf],[Nf_n]\}-\min\{[Nf],[Nf_n]\}\leq 2$．于是设$M=\min\{[Nf],[Nf_n]\}$，则$M$关于$n>n_1$，一致有界且

$$\int_0^N\{[|y|f_n]-[|y|f]\}\ d\Omega_n(y)=\sum_{j=0}^{M}j\{\int_{\frac{j}{f_n}}^{\frac{j}{f}}d\Omega_n(y)+\int_{\frac{j+1}{f}}^{\frac{j+1}{f_n}}d\Omega_n(y)\}+$$

$$\sum_{j=M+1}^{\max\{[Nf],[Nf_n]\}}\left|\int_{\frac{j}{f_n}}^{\frac{j+1}{f_n}}[yf_n]\ d\Omega_n(y)(\text{或}\int_{\frac{j}{f}}^{\frac{j+1}{f}}[yf]\ d\Omega_n(y))+O(\frac{1}{N})\ .\right. \tag{1.13}$$

注意到$\Omega_n(y)\to\Omega(y)(n\to\infty)$对每一个$y$成立，则对任$\delta>0$，当$n>n_2$时

$$\left|\sum_{j=0}^{M}j\{\int_{\frac{j}{f_n}}^{\frac{j}{f}}\cdot+\int_{\frac{j+1}{f}}^{\frac{j+1}{f_n}}\cdot\}\right|\leq 4\sum_{j=0}^{M}j\{\left|\int_{\frac{j}{f}+\delta j}^{\frac{j}{f}}\cdot\right|+\int_{\frac{j+1}{f}}^{\frac{j+1}{f}+j\delta}\cdot\}=o(1) \tag{1.14}$$

另一方面，显然

$$\int_N^{\infty}[|y|f_n]\, d\Omega_n(y)\le f_n\frac{1}{N},\quad \int_N^{\infty}[|y|f]\, d\Omega_n(y)\le f\frac{1}{N}. \tag{1.15}$$

（1.11）－（1.15）表明$\lim\limits_{n\to\infty}\int_0^{\infty}[|y|f_n]\,d\Omega_n(y)=\int_0^{\infty}[|y|f]\,d\Omega(y)$．

同理$\lim\limits_{n\to\infty}\int_{-\infty}^{0}[|y|f_n]\,d\Omega_n(y)=\int_{-\infty}^{0}[|y|f]\,d\Omega(y)$．

于是（1.10）成立．从而$\lim\limits_{n\to\infty}c(\psi,L_n,x)=1+\frac{2}{\sqrt{2\pi}}\int_0^{\infty}[y(\frac{\varphi}{\psi})^{1/2}]\,e^{-y^2/2}dy$.

由此不难得到（1.6）．定理1证毕．

§2 应用

1）Gamma－型算子的渐近常数

Gamma－型算子是指

$$G_n^r(f;\ x)=\frac{1}{(n+r)!}(\frac{n}{x})^{n+r+1}\int_0^{\infty}f(u)\ u^{n+r}e^{-\frac{nu}{x}}du,$$

其中r是整数，不难验证[3]

$$T_{n,j+1}(x)=\frac{x}{n}\{(r+j+1)\ T_{n,j}(x)+jxT_{n,j-1}(x)\}. \tag{2.1}$$

因此$T_{n,1}(x)=\frac{x(r+1)}{n}$，$T_{n,2}(x)=\frac{x^2}{n}(1+\frac{r+1}{n})$．由（2.1）知$T_{n,j}(x)$满足定理1的条件（1.3)(1.4）．若记$\varphi_n(x)=x^2(1+\frac{r+1}{n})$．则$\lim\limits_{n\to\infty}\varphi_n(x)=x^2$．于是我们重新得到了定理R—S.

2）指数型算子的渐近常数．

指数型算子的定义是[6]

$M_n(f;\ x)=\int_A^B f(u)\ W(n,x,u)\ du$，其中$W(n,x,u)$满足：

i) $W(n,x,u)\ge 0$；ii) $\int_A^B W(n,x,u)\ du=1$；iii) $\frac{\partial}{\partial x}W(n,x,u)=\frac{n}{\varphi(x)}(u-x)(W(n,x,u)$，这里$\varphi(x)$是阶不高于2的代数多项式，当$x\in(A,B)$时$\varphi(x)>0$，若$A,B\ne\pm\infty$，则$\varphi(A)=\varphi(B)=0$.

易证[5]

$$T_{n,k}(x)=\sum_{j=1}^{[\frac{k}{2}]}b_{n,k,j}(\frac{\varphi}{n})^j(\frac{\varphi'}{n})^{\delta_k}, \tag{2.2}$$

其中$\delta_{2\nu}=0$，$\delta_{2\nu+1}=1$，$b_{n,2k,j}=C_{n,2k,j}\frac{1}{n^{2(k-j)}}$, $b_{n,2k-1,j}=C_{n,2k+1,j}\frac{1}{n^{2(k-1)}}$，$C_{n,2k,k}=(2k-1)!!+O(\frac{1}{n})$.记$c_{n,m}=\max\limits_j|c_{n,m,j}|$，则对适当的常数$D$, $c_{n,m}\le D^m\cdot m!$.由此，不难证实$T_{n,k}(x)$满足（1.3)(1.4).并且$T_{n,2}(x)=\frac{\varphi(x)}{n}$．于是有

定理2[5] $\lim\limits_{n\to\infty}n^{-a/2}E(a,M,\ M_n,\ x)=\varphi^{a/2}(x)\frac{\Gamma(\frac{1+a}{2})}{\pi^{1/2}}2^{a/2}\cdot M$,

$$\lim_{n\to\infty}c(\psi,\ M_n,\ x)=2\sum_{j=0}^{\infty}(j+1)\{\Phi((\frac{\psi}{\varphi})^{1/2}(j+1))-\Phi((\frac{\psi}{\varphi})^{1/2}j)\}.$$

3）Phillips算子的渐近常数

Phillips算子是

$$S_\lambda(f;\ x)=\int_0^{\infty}f(u)\sum_{n=1}^{\infty}\frac{(\lambda^2x)^n}{n!}\frac{u^{n-1}}{(n-1)!}du+f(o)e^{-\lambda x},$$

May[7] 指出Phillips算子不是指数型算子．现在我们利用定理1来求其渐近常数．已知[7] $T_{\lambda,1}(x)=0$ ，$T_{\lambda,2}(x)=\frac{2x}{\lambda}$，一般地

$$T_{\lambda,j+1}(x)=\frac{2xj}{\lambda}T_{\lambda,j-1}(x)+\frac{2x}{\lambda}T'_{\lambda,j}(x)+\frac{x}{\lambda^2}T''_{\lambda,j}(x)+\frac{xj(j-1)}{\lambda^2}T_{\lambda,j-2}(x)+\frac{2xj}{\lambda^2}T'_{\lambda,j-1}(x) \qquad (2.3)$$

我们来验证$T_{\lambda,j}(x)$ 满足（1.3)(1.4）．首先，从（2.3）易得$T_{n,k}(x)$ 是设有常数项的〔$\frac{k}{2}$〕次多项式．设$T_{\lambda,k}(x)=\sum_{j=1}^{[\frac{k}{2}]}b_{\lambda,k,j}x^j$，

那么，不难用归纳法证明：

$$b_{\lambda,2k,j}=O(\frac{1}{\lambda^k})\ ,\quad b_{\lambda,2k+1,j}=O(\frac{1}{\lambda^{k+1}})\ . \qquad (2.4)$$

事实上，$b_{\lambda,2,j}=O(\frac{1}{\lambda})$ ，$b_{\lambda,3,j}=O(\frac{1}{\lambda^2})$ ．假如（2.4）对$k\leq N$成立．那么，由（2.3）我们看到（2.4）当$k=N+1$时也成立．现在容易证实$T_{\lambda,j}(x)$ 满足（1.3)(1.4）．因为由（2.3）(2.4)

$$\frac{T_{\lambda,2k+2}(x)}{T^{k+1}_{\lambda,2}(x)}=\frac{\frac{2x}{\lambda}(2k+1)}{T_{\lambda,2}(x)}\frac{T_{\lambda,2k}(x)}{T^k_{\lambda,2}(x)}+O(\frac{1}{\lambda})=(2k+1)\frac{T_{\lambda,2k}(x)}{T^k_{\lambda,2}(x)}+O(\frac{1}{\lambda})\ ,$$

$$\frac{T_{\lambda,2k+1}(x)}{T^{k+\frac{1}{2}}_{\lambda,2}(x)}=O(\lambda^{-1/2})\ .$$

因此，我们得到

定理3 $\lim\limits_{n\to\infty}\lambda^{-a/2}E(a,\ M,\ S_\lambda,\ x)=2^a\frac{\Gamma(\frac{1+a}{2})}{\pi^{1/2}}Mx^{a/2}$,

$$\lim_{n\to\infty}c(\psi,\ S_\lambda,\ x)=2\sum_{j=0}^{\infty}(j+1)\ \{\Phi(\sqrt{\frac{\psi(x)}{2x}}(j+1))-\Phi(\sqrt{\frac{\psi(x)}{2x}}j)\}\ .$$

参 考 文 献

〔1〕C.G.Esseen,Numer.Math.2( ) ,20 —21 .
〔2〕R.K.S.Rathore,Math.Z.,141(1975) ，193 —198 .
〔3〕R.K.S.Rathore and O.P.singh,Acta.Math.Sci.Hung.,36(1—2)(1980) ，155—160.
〔4〕P.C.Sikkema,Numer.Math.,1(1959) ，221—239.
〔5〕周信龙，指数型算子的渐近常数(已投科学通报）．
〔6〕K.Sato,J.Appro.T.,32(1981) ，32—46.
〔7〕C.P.May,J.Appro.T.,20(1977) ，315—332.

On the Asymptotic Constants for acclass of Positive Linear Operators

Zhou Xinlong
(Hangzhou University)

The ppsitive linear operators are defined by

$$L_n(f, x) \int_A^B f(u)\, d\sigma_{n,x}(u), \quad \forall f \in C[A, B],$$

for which, the following conditions are satisfied

$$\int_A^B d\sigma_{n,x}(u) = 1, \quad \lim_{n\to\infty} L_n(f; x) = f(x), \quad \forall x \in C[A, B].$$

Let $T_{nj}(x) = \int_A^B (u - x)^j d\sigma_{n,x}(u)$, $\quad w(f; \delta) = \sup_{\substack{|x-y|\le\delta \\ x, y \in [A, B]}} |f(x) - f(y)|$.

$$\mathrm{Lip}_M \alpha = \{ f \in C[A, B];\ w(f;\delta) \le M\delta^\alpha \}, \quad E(\alpha, M, L_n, x) = \sup_{f \in \mathrm{Lip}_M \alpha} |L_n(f, x) - f(x)|.$$

$$c(\psi, L_n, x) = \inf\{ c:\ |L_n(f; x) - f(x)| \le c w\left(f; \left(\frac{\psi(x)}{n}\right)^{1/2}\right) \}.$$

where $\psi(x) \ge 0$, $\psi(x) \in c[A, B]$.

In this paper, we prove the following theorem

theorem If

$$\lim_{n\to\infty} \frac{T_{n2j}(x)}{T_{n,2}^j(x)} = (2j-1)!!, \quad (j = 1, 2, \cdots,); \quad \lim_{n\to\infty} \frac{T_{n,2j+1}(x)}{T_{n,2}^{j+1/2}(x)} = 0, \quad (j = 0, 1, 2, \cdots,).$$

Then $\lim_{n\to\infty} T_{n,2}^{-\alpha/2}(x)\, E(\alpha, M, L_n, x) = M \dfrac{\Gamma\left(\frac{1+\alpha}{2}\right)}{\pi^{1/2}} 2^{\alpha/2}$.

Suppose that $T_{n,2}(x) = \dfrac{\varphi_n(x)}{n}$, $\lim_{n\to\infty} \varphi_n = \varphi$ and x is not the zero of φ and ψ. Then

$$\lim_{n\to\infty} c(\psi, L_n, x) = 2 \sum_{j=0}^{\infty} (j+1) \left\{ \Phi\left(\left(\frac{\psi}{\varphi}\right)^{1/2} (j+1) \right) - \Phi\left(\left(\frac{\psi}{\varphi}\right) j \right) \right\},$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt.$$