J. Math. Res. & Exposition, 1986, No. 3

Solution to a Problem of J.P.Hutchinson and P.B.Trow*

W. D. Wei (魏万迪)
Department of Mathematics
Sichuan University and
Department of Computer Science,
University of Illinois at Urbana-Champaign

C. L. Liu
Department of Computer Science
University of Illinois at Urbana-Champaign

Use of the Pigeonhole Principle to solve the following problem was discussed in several books, for example [1,2]. Suppose a player (e. g., tennis or chess) practices on d consecutive days, playing at least one game a day and a total of no more than b games where $d<b<2d$. Then we can assert that for each $i\leqslant 2d-b-1$, there are some consecutive days on which exactly i games are played totally. In [3], J. P. Hutchinson and P.B. Trow investigated this problem in detail. They also posed an unsettled question to which we provide an answer in this note.

We follow the tetminology in [3]. Let d, b $(d<b)$ be two fixed positive integers. Let $a=(a_1, \cdots, a_d)$ be a strictly increasing sepuence of positive integers. Then a is called an admissible sum sepuence if $a_d\leqslant b$. An admissible sum sequence is said to have property k, k a positive integer, if either an entry of the sequence equals k or two entries differ dy k. Let A be the set of integers i such that every admissible sum sequence has property i, and A^c the complement of A.

The question posed dy J. P. Hutchinson and P. B. Trow is as follows: "Is there, in general, a simple characterization of the least integer in A^c?" And the they[3] and Wang and Wu[4] settled this problem for the special cases $d<b<\frac{3}{2}d$ and $\frac{3}{2}d\leqslant b<\frac{7}{4}d$, respectively. In the present note, We provide an answer to this problem for the general case.

We summarize some of the results in [3] Which we need later on:

Lemma 1 [3,Th.2]: Given positive integers d, b $(d<b)$ and k, k is in A if and only if $2d-r>b$, where r is the least non-negative residue of d, mod k.

Lemma 2 [3,Cor.3]: If $b\geqslant 2d$, A is empty.

According to Lemma 2, we assume that $b<2d$ from now on.

* Received Jan. 4, 1984.

Lemma 3 [3, Th. 4]: $\{1, 2, \cdots, 2d-b\} \in A$.

Theorem: Let b, d $(d<b<2d)$ be two given positive integers, and r_i $(i \geqslant 0)$ be the least non-negative residue of $b-d$, mod $2d-b+1+i$. Then the least integer in A^c is $2d-b+i_0+1$, where $i_0 = \min\{i \mid i \geqslant r_i\}$.

Proof By Lemma 1, for given k, k is in A^c if and only if $r \geqslant 2d-b$, where r is the least non-negative residue of d, mod k. That is, k is in A^c if and only if there is an integer r such that $k > r \geqslant 2d-b$ and $k \mid d-r$. Let $k = 2d-b+i_0+1$, $r = 2d-b+r_{i_0}$. By the definition of i_0 and r_{i_0}, $k > 2d-b+i_0 \geqslant 2d-b+r_{i_0} = r \geqslant 2d-b$; $d-r = d-(2d-b+r_{i_0}) = b-d-r_{i_0} \equiv 0 \pmod{k}$. Hence $k \in A^c$.

Now we prove that $l \in A$ for all l, $1 \leqslant l \leqslant 2d-b+i_0$. by Lemma 3, we need only to consider the l's, betwen $2d-b+1$ and $2d-b+i_0$. For such l, let $h = l - (2d-b)$. Suppose there is an integer r such that $l > r \geqslant 2d-b$ and $l \mid d-r$. Let $j = r-(2d-b)$. Then

$$0 \leqslant j < h \leqslant i_0,\ l = 2d-b+1+(h-1); \qquad b-d = d-r+j \equiv j \pmod{(2d-b+1+(h-1))}.$$

But $j \leqslant h-1 < i_0$, a contradiction to the definition of i_0.

The theorem provides a simple algorithm, which is based only on the division algorithm. By this algorithm, the least number in A^c can be found easily in the general case.

Here are two illustrative examples.

Example 1 Suppose $d = 5000$, $b = 9000$. Then $2d-b+1 = 1001$ and $b-d = 4000 = 3 \cdot 1001 + 997 = 3(1001+i) + 997 - 3i$. Since the minimal value of i satisfying

$$0 \leqslant 997 - 3i \leqslant i,\ i.e.,\ \frac{997}{4} \leqslant i \leqslant \frac{997}{3}$$

exists and epuals to 250, the least number in A^c in this case is $1001 + 250 = 1251$.

Example 2 Suppose $d = 5000$, $b = 9990$. Then $2d-b+1 = 11$ and

$$b-d = 4990 \equiv 7 \pmod{11} \equiv 10 \pmod{12} \equiv 11 \pmod{13} \equiv 6 \pmod{14} \equiv 10 \pmod{15}$$
$$\equiv 14 \pmod{16} \equiv 9 \pmod{17} \equiv 4 \pmod{18}.$$

Since $7>0, 10>1, 11>2, 6>3, 10>4, 14>5, 9>6, 4<7$, the least number in A^c in this case is 18.

References

[1] Liu, C. L., Elements of Discrete Mathematics, Mcgraw-Hill, New York, 1977.

[2] Brualdi, R. A., Introductory Combinatorics, North-Holland, New York, 1977.

[3] Hutchinson, J.P., and P.B. Trow, Some Pigeonhole Principle Results extented, Amer. Math. Monthly 87(1980), 648—651.

[4] Wang Yufeng and Wu Leguang, A Note on Pigeonhole Principle, J. Math. Research and Exposition, 1981, No. 2, 103-106.