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Mistakes in the paper "New proofs of some theorems on infinitely differentiable functions"[1]*

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Abstract

In the paper [1], the author told us that he provides new proofs for the three classical theorems on real singularity theory: Mather's two theorms, and the splitting lemma. And in the paper [2], the author used this method to catastrophe theory.

The journal "Mathematical Reviews" ever commented the above papers successivelly in Vol.57, No.3 (1979) and Vol.58, No.3 (1979).

This paper points out that the proofs about Mather's theorems I and II in [1] are wrong, and then presents an counter-example to show that the corollary and conjecture are false.

Fistly, in the new proof of Mather's theorem I (PP. 166-167) the existence of $n_i^0(x)$, $n_i^1(x)\cdots\cdots((18)-(20))$ is constructed term by term. Then the following formula (17) is induced from the well-known Borel lemma:

$$n_i(x,t)=\sum_{s=0}^{\infty}n_i^{(s)}(x)t^s \tag{17}$$

Here some questions arise: If this expression is only a pure formal symbol, this is, $n_i(x,t)$ stands only for the formal power series $\sum_{s=0}^{\infty}n_i^{(s)}(x)t^s$, then if it is disconvergent the differential equation in the sequel (P.167) can't be obtained, and there is nothing about smooth solution $H(x,t)$(P.167). On the other side, if $n_i(x,t)$ is, by Borel lemma, the smooth function corresponding to the formal power series $\sum_{s=0}^{\infty}n_i^{(s)}(x)t^s$, then we could only say: "$n_i(x,t)$ possesses $\sum_{s=0}^{\infty}n_i^{(s)}(x)t^s$ as its Taylor series". We know that the equality

$$n_i(x,t)=\sum_{s=0}^{\infty}n_i^{(s)}(x)t^s$$

means that "$n_i(x,t)$ is analytic". But, however, it is impossible from Borel

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lemma. Furthermore, if the formal power series is disconvergent (*i.e.* the equality is not true.) then the mapping H produced from formula (21) does not satisfy formula (16), because $n_i(x,t)$ is "only a function" but not a sum of the terms $n_i^{(0)}(x)$, $n_i^{(1)}(x)t$, …. Thus the procedure (18)—(20) can not be returned back to (16). Therefore formula (23) can't be set up in that way.

Secondly, the following example will make it clear that the proof of Mather's theorem Ⅱ, the corollary and the conjecture are all not correct.

Example Suppose f, g: $\mathbf{R}\to\mathbf{R}$ is defined as below:

$$f(x)=x^2+x^3,\qquad g(x)=x^2-x^3$$

It is easy to see that $f(x)$ is 2-determined from both the classical method of finite determinacy" and the method of Deakin (since $h(x)=x^k$ is k-determined). The function $f(x)$ has the same 2-jet as $g(x)$. (This is the notation $^2f(x)={}^2g(x)=x^2$ in [1]). Hence f $\underset{\sim}{R}$ g. Then there exists a transform of coordinates ξ such that

$$g=f\circ\xi$$

In fact, it is easy to see that

$$\xi(x)=-x,$$

and this transform is unique. On the other side, if there is another transform:

$$\xi_1(x)=ax+bx^2+o(x^2)$$

then ($g=f\circ\xi_1$)

$$g(x)=a^2x^2+a^3x^3+o(x^4)$$

when $x\to 0$. Comparing this expression of $g(x)$ with its definition

$$g(x)=x^2-x^3$$

we obtain $a^2=1$, $a^3=-1$, hence $a=-1$, $\xi_1(x)=-x+bx^2+o(x^2)$, and so $b=0$ $o(x^2)=0$. Finally, such a transform $\xi(x)=x+o(x)$ is impossible. Thus the corollary $f(x)\ \underset{\sim}{Q}\ g(x)$ (P.170) and the conjecture (P.173) are not correct. While "Let $\xi^{(0)}=x$" in the proof (P.169) is legalless.

References

[1] Michael A. B. Deakin, New proofs of some theorems on infinitely differentiable functions, Bull. Austral. Math. Soc. 17(1977), No. 2, 161—175.

[2] ------, An elementary approach to catastrophe theory, Bull. Math. Biol. 40 (1978) No. 4, 429—450.

[3] Mathematical Reviews, Vol. 57 No. 3(1979) 1004—1005 MR 57#7650.

[4] -----, Vol. 58 No. 3(1979) P. 1984, MR 58#13135.