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On an Extension of a Partition Identity and Its Abel-Analog*

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Recently, Hwang & Wei (1983) have proved the following partition identity:

Proposition. Let $n_i (1 \leq i \leq m)$ be integers and $\sum_{i=1}^{m} n_i = n$. Then

$$\sum_{k_1+k_2+\cdots+k_m=k} \prod_{i=1}^{m} \binom{n_i+1-k_i}{k_i} = \sum_{j\geq 0} \binom{j+m-2}{m-2} \binom{n+1-k-2j}{k-2j} \quad (1)$$

The combinatorial interpretation of the number $\binom{n-k+1}{k}$ and the induction principle were their main means. By using the power series method of G Gould (1984), this note will presents a multi-fold extension to (1) and its Abel's coefficient analog.

I. Generalization

For any complex numbers a, b, and c, Gould (1984) has found the following general closed form.

Lemmaa I. I. $$\sum_{k} \frac{a+ck}{a+(b+c)k} \binom{a+(b+c)k}{k} z^k = x^a \cdot \frac{x-b(x-1)}{x-(b+c)(x-1)} \quad (2)$$

where $z = (x-1)x^{-(b+c)}$.

From this we can prove a generalization to (1).

Theorem I. 2. Let a_i be complex numbers and $a_1+a_2+\cdots+a_m = a$.

$$\sum_{k_1+k_2+\cdots+k_m=k} \prod_{i=1}^{m} \frac{a_i+ck_i}{a_i+(b+c)k_i} \binom{a_i+(b+c)k_i}{k_i}$$

$$= \sum_{i,j} \binom{m}{i} \binom{i-1+j}{j} c^i (b+c)^j \cdot \frac{a+(i+j)(b+c-1)}{a+(b+c)k-i-j} \binom{a+(b+c)k-i-j}{k-i-j} \quad (3)$$

where we take $\binom{-1}{0} = 1$.

Proof. From (2) we know that the left side of (3) is equal to the coefficient of z^k in the expansion of the following fraction

$$x^a \left[\frac{x-b(x-1)}{x-(b+c)(x-1)} \right]^m \quad (4)$$

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According to $\frac{x-b(x-1)}{x-(b+c)(x-1)}=1+\frac{c(x-1)}{x-(b+c)(x-1)}$ and Lemma1.1, we have the following expansion to (4):

$$x^a \sum_{i,j} \binom{m}{i} \binom{i-1+j}{j} \cdot c^i (b+c)^j \left(\frac{x-1}{x}\right)^{i+j}$$

$$= \sum_{i,j} \binom{m}{i} \binom{i-1+j}{j} \cdot c^i (b+c)^j x^{a+(b-c-1)(i+j)} z^{i+j} \tag{5}$$

Comparing (2)and (5) we have (3) and the theorem is proved.

If we take the following expansion instead of (5),

$$x^a \frac{x-b(x-1)}{x-(b+c)(x-1)} \sum_{i,j} \binom{m-1}{i} \binom{i-1+j}{j} c^i (b+c)^j \left(\frac{x-1}{x}\right)^{i+j}$$

$$= \sum_{i,j} \binom{m-1}{i} \binom{i-1+j}{j} c^i (b+c)^j x^{a+(b+c-1)(i+j)} \frac{x-b(x-1)}{x-(b+c)(x-1)} z^{i+j}$$

we obtain an alternate form of (3):

$$\sum_{k_1+k_2+\cdots+k_m=k} \prod_{i=1}^{m} \frac{a_i+ck_i}{a_i+(b+c)k_i} \binom{a_i+(b+c)k_i}{k_i}$$

$$= \sum_{i,j} \binom{m-1}{i} \binom{i-1+j}{j} c^i (b+c)^j \cdot \frac{a+ck+(i+j)(b-1)}{a+(b+c)k-i-j} \binom{a+(b+c)k-i-j}{k-i-j} \tag{6}$$

If we take $c=0$ and $b=0$ in (3) and (6), respectively, we have the following degenarate consequences:

Corollary I.3.

$$\sum_{k_1+k_2+\cdots+k_n=k} \prod_{i=1}^{m} \frac{a_i}{a_i+k_ib} \binom{a_i+k_ib}{k_i} = \frac{a}{a+kb} \binom{kb+a}{k} \tag{7}$$

Notice that $\sum_{i,j} \binom{m-1}{i} \binom{i-1+j}{j} c^{i+j} = \sum_n \binom{m+n-2}{n} c^n$, we have the multifold extension of Jensen's formula due to Gould (1960).

Corollary I.4. $\sum_{k_1+k_2+\cdots+k_m=k} \prod_{i=1}^{m} \binom{a_i+k_ic}{k_i} = \sum_n \binom{m+n-2}{n} \binom{a+kc-n}{k-n} c^n$

(8)

If we take $c=-1$, and $a_i=n_i+1$, then the left side of (8) equals

Coef. of t^k in $\frac{1}{(1+t)^{m-1}(1-t)^{n+m-2k+1}}$ = Coef. of t^k in $\frac{1}{(1-t^2)^{m-1}(1-t)^{n-2k+2}}$

$$= \sum_{j\geq 0} \binom{j+m-2}{m-2} \binom{n-k-2j+1}{k-2j}.$$

This is the result (1) of Hwang & Wei (1983).

Ⅱ. Abel-Analog

In this section, we study the Abel's coefficient analog to the results mentioned in section I. The all conditions and proofs are quite the same as section I.

Lemma 2.1 (Gould, 1984).

$$\sum_k \frac{a+ck}{a+(b+c)k} \frac{[a+(b+c)k]^k}{k!} \cdot z^k = x^a \cdot \frac{1-b\log x}{1-(b+c)\log x}, \tag{9}$$

where $z = x^{-(b+c)} \log x$.

From this, the following theorem can be proved.

Theorem 2.2.

$$\sum_{k_1+k_2+\cdots+k_m=k} \prod_{i=1}^{m} \frac{a_i+ck_i}{a_i+(b+c)k_i} \frac{[a_i+(b+c)k_i]^{k_i}}{k_i!}$$

$$= \sum_{i,j} \binom{m}{i} \binom{i+j-1}{j} c^i \cdot \frac{a+(b+c)(i+j)}{a+k(b+c)} \cdot \frac{[a+(b+c)k]^{k-i-j}}{(k-i+j)!} (b+c)^j \tag{10}$$

Alternatively,

$$\sum_{k_1+k_2+\cdots+k_m=k} \prod_{i=1}^{m} \frac{a_i+ck_i}{a_i+(b+c)k_i} \frac{[a_i+(b+c)k_i]^{k_i}}{k_i!}$$

$$= \sum_{i,j} \binom{m-1}{i} \binom{i-1+j}{j} c^i (b+c)^j \cdot \frac{a+ck+(i+j)b}{a+(b+c)k} \frac{[a+(b+c)k]^{k-i-j}}{(k-i-j)!} \tag{11}$$

As theorem 1.2, it has the following consequences:

Corollary 2.3.

$$\sum_{k_1+k_2+\cdots+k_m=k} \prod_{i=1}^{m} \frac{a_i}{a_i+bk_i} \frac{(a_i+bk_i)^{k_i}}{k_i!} = \frac{a}{a+bk} \frac{(a+bk)^k}{k!} \tag{12}$$

Corollary 2.4.

$$\sum_{k_1+k_2+\cdots+k_m=k} \prod_{i=1}^{m} \frac{(a_i+ck_i)^{k_i}}{k_i!} = \sum_n \binom{m+n-2}{n} \frac{(a+ck)^{k-n}}{(k-n)!} c^n \tag{13}$$

The last one can be seen as a multi-fold extension of Jensen's formula for Abel's coefficient[1].

References

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[3] H. W. Gould, New inverse series relations for finite and infinite series with applications, J. Math. Research & Exposition, 4(1984)(2), 119-130.