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On a Theorem of Milin

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Let S be class of functions $f(z)=z+a_2+z^2+\cdots$ analytic and univalent in the unit disk D, and let $F_\lambda(z)=\{\frac{f(z)}{z}\}^\lambda=1+\sum_{n=1}^{\infty}D_n(\lambda)z^n$. In [1] I.M.Milin proved that

Theorem A. If $f\in S$ and $\lim_{\rho\to1}\frac{(1-\rho)^2}{\rho}|f(\rho)|=\alpha\neq0$, then

$$\lim_{n\to\infty}\frac{|D_n(\lambda)-D_{n-1}(\lambda)|}{d_n(2\lambda-1)}\leq1,\quad \frac{3}{4}<\lambda<1, \tag{1}$$

where $d_0(h)=1$ and $d_n(h)=\frac{h(h+1)\cdots(h+n-1)}{n!}$.

In fact, the result is deduced from Milin's Tauberian Theorem. Here we give another proof of (1) with aid of the following theorem.

Theorem B.[2,3] Let $\omega(z)=\sum_{k=1}^{\infty}A_kz^k$, $\varphi(z)=e^{\omega(z)}=1+\sum_{k=1}^{\infty}D_k^{(1)}z^k$. If

(a) $h^{-p}\sum_{k=1}^{\infty}k^{p-1}|A_k|^px^k=\log\frac{1}{1-x}+C+o(1)$, $x\to1$, where C is a constant; and

(b) $\left|\frac{D_m^{(1)}}{d_m(t)}\right|^p-\left|\frac{D_n^{(1)}}{d_n(t)}\right|^p\to0$, as $n\to\infty$ and $\frac{m}{n}\to1$. Then

$$\lim_{n\to\infty}\frac{|D_n^{(1)}|}{d_n(h)}\leq e^{Cp}. \tag{2}$$

To deduce (1) from Theorem A. we set $p=2$ and $\theta_0=0$, the direction of maximal growth of $f(z)$. Let

$$\omega_1(z)=\log(1-z)F_\lambda(z)=\log(1-z)\{\frac{f(z)}{z}\}^\lambda=\sum_{k=1}^{\infty}(2\lambda r_k-\frac{1}{k})z^k=\sum_{k=1}^{\infty}A_k^{(1)}z^k$$

and $\varphi_1(z)=(1-z)F_\lambda(z)=1+\sum_{k=1}^{\infty}[D_k(\lambda)-D_{k-1}(\lambda)]z^k=\sum_{k=1}^{\infty}D_k^{(1)}z^k$, $D_0=1$, (3)

Bazilevich Theorem[1] asserts that

$$\sum_{k=1}^{\infty}k\left|r_k-\frac{1}{k}\right|^2\leq\frac{1}{2}\log\frac{1}{\alpha}\quad(\alpha\neq0) \tag{4}$$

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Thus, using the condition of Theorem A to deduce

$$\sum_{k=1}^{\infty} k\,|A_k^{(1)}|^2 x^k = \sum_{k=1}^{\infty} k\,|2\lambda r_k - \frac{1}{k}|^2 x^k = 4\lambda^2 \sum_{k=1}^{\infty} k\,|r_k - \frac{1}{k}|^2$$

$$+ 2\lambda(2\lambda - 1)\log\frac{(1-x)^2}{x}|f(x)| + (2\lambda - 1)^2 \log\frac{1}{1-x}$$

$$= (2\lambda - 1)^2 \log\frac{1}{1-x} + C + o(1). \quad (C \le 2\lambda(1-\lambda)\log a) \qquad (5)$$

The condition (a) has been verified for $h = 2\lambda - 1$. To prove (1) it is sufficient to verify condition (b). we need the following lemma.

Lemma. Let $\theta_0 = 0$ be the direction of maximal growth of $f(z)$ and $\lim\limits_{\rho \to 1}(1-\rho)^2|f(\rho)| = a > 0$. Then

$$J_2 = \int_0^{2\pi} |1 - re^{i\theta}|^2 |\frac{f(re^{i\theta})}{r}|^{2\lambda} d\theta \le \frac{A}{(1-r)^{4\lambda - 3}}, \quad 1 \ge \lambda > \frac{3}{4} \qquad (6)$$

where A is a constant only dependent on a and λ.

Proof. By Goluzin inequality

$$\sum_{i,j=1}^{n} \gamma_i \bar{\gamma}_j \log\frac{f(z_i) - f(z_j)}{z_i - z_j} \cdot \frac{z_i z_j}{f(z_i) f(z_j)} | \le \sum_{i,j=1}^{n} \gamma_i \bar{\gamma}_j \log\frac{1}{1 - z_i \bar{z}_j},$$

taking $n = 2$, $\gamma_1 = 1$, $\gamma_2 = \varepsilon (> 0)$ and $z_1 = \rho$, $z_2 = z = re^{i\theta} (r = \rho^2)$, we get

$$1 \le |\frac{f(\rho) - f(z)}{\rho - z}|^{2\varepsilon} |\frac{\rho z}{f(\rho) f(z)}|^{2\varepsilon} |\rho \frac{f'(\rho)}{f^2(z)}| |z \frac{f'(z)}{f^2(z)}|^{\varepsilon^2} \cdot$$

$$\frac{1}{(1-r^2)^{\varepsilon^2}(1-\rho^2)|1 - r\rho e^{i\theta}|^{2\varepsilon}} \qquad (7)$$

It is known that $a \le \frac{(1-\rho)^2}{\rho}|f(\rho)|$, $|z\frac{f'(z)}{f(z)}| \le \frac{1+|z|}{1+|z|}$, and $\frac{1}{|1 - r\rho e^{i\theta}|} \le \frac{2}{|1 - re^{i\theta}|}$, $\frac{1}{|\rho - re^{i\theta}|} \le \frac{2}{\sqrt{r}\,|1 - re^{i\theta}|} (r = \rho^2)$. Substituting this result into (7) yields

$$|f(re^{i\theta})| \le \frac{A}{|1 - re^{i\theta}|^{2-\delta}(1-r)^{\delta'}}, \quad \delta = \frac{2\varepsilon}{2+\varepsilon}. \qquad (8)$$

Choose δ so small that $4\lambda - 3 > 2\delta\lambda > 0$, Thus we conclude

$$J_2 \le \frac{A}{(1-r)^{2\delta\lambda}} \int_0^{2\pi} \frac{dB}{|1 - re^{i\theta}|^{4\lambda - 2 - 2\delta\lambda}} \le \frac{A_1}{(1-r)^{4\lambda - 3}}.$$

Corollary. With the above assumption, we have

$$\int_0^{2\pi} |1 - re^{ik\theta}| |\frac{f(re^{i\theta})}{r}|^{\lambda} d\theta \le \frac{A}{(1-r)^{2\lambda - 3/2}} k^{1/2} \qquad (9)$$

for some k.

Proof. Applying Cauchy's inequality to the left side of (9), we get

$$\int_0^{2\pi} |\frac{r^k e^{ik\theta} - 1}{re^{i\theta} - 1}| |1 - re^{i\theta}| |\frac{f(z)}{z}|^{\lambda} d\theta \le$$

$$\sqrt{\int_0^{2\pi} |\frac{r^k e^{ik\theta} - 1}{re^{i\theta} - 1}|^2 d\theta \int_0^{2\pi} |1 - re^{i\theta}|^2 |\frac{f(z)}{z}|^{2\lambda} d\theta} \le Ak^{1/2}(1-r)^{-2\lambda + 3/2}, \; z = re^{i\theta}$$

by Lemma.

we now come to verify (b). By (4) we have

$$\sum_{k=1}^{\infty} k^2|r_k-\frac{1}{k}|^2x^k=o(\frac{1}{1-x}),\quad x\to 1-o. \tag{10}$$

Let $t_n=n(D_n(\lambda)-D_{n-1}(\lambda))$, $k=n-m(>0)$, and $\psi(z)=\frac{f(z)}{z}$, then $F'(z)=$
$=\lambda(1-z)\psi^\lambda\{\frac{\psi'}{\psi}-\frac{2}{1-z}\}+(2\lambda-1)\psi^\lambda$. Therefore

$$t_n=\frac{1}{2\pi r^n}\int_0^{2\pi}F_\lambda'(re^{i\theta})e^{-i(n-1)\theta}d\theta \tag{11}$$

and

$$|t_n-t_m|\le\frac{1}{2\pi r^n}\{\lambda\int_0^{2\pi}|1-r^ke^{ik\theta}||1-re^{i\theta}||\psi|^\lambda|\frac{\psi'}{\psi}-\frac{2}{1-z}|d\theta$$

$$+(2\lambda-1)\int_0^{2\pi}|1-r^ke^{ik\theta}||\psi|^\lambda d\theta\}\le\frac{1}{2\pi r^n}\{2\lambda J_1+(2\lambda-1)J_2\}. \tag{12}$$

Applying Cauchy's inequality

$$J_1=\int_0^{2\pi}|1-re^{i\theta}||\psi^\lambda||\frac{\psi'}{\psi}-\frac{2}{1-z}|d\theta\le(\int_0^{2\pi}|1-re^{i\theta}|^2|\psi^\lambda|^2d\theta)\int_0^{2\pi}|\frac{\psi'}{\psi}$$

$$-\frac{2}{1-re^{i\theta}}|^2d\theta=\sqrt{J_{1,1}J_{1,2}}$$

In virtue of (10) and (9), we have

$$J_{1,2}=2\pi\sum_{k=1}^{\infty}k|r_k-\frac{1}{k}|^2r^{2k}=o(\frac{1}{1-r}).\ r\to 1-o,\ \text{and}\ J_{1,1}\le\frac{A}{(1-r)^{4\lambda-3}}.$$

It follows that

$$J_1=o(\frac{1}{(1-r)^{2\lambda-1}}). \tag{13}$$

Inequalities (9), (12) and (13) in combination yield

$$t_n-t_m=o(n^{1-2\lambda}) \tag{14}$$

for $r=1-\frac{1}{n}$, $n\to\infty$, $\frac{m}{n}\to 1$.

If $k=0$, it is easy to deduce $t_n=o(n^{1-2\lambda})$ from (11) because

$$\int_0^{2\pi}|\frac{f(z)}{z}|^\lambda d\theta\le\frac{A}{(1-r)^{2\lambda-1}},\ z=re^{i\theta},\ J_1=o(\frac{1}{(1-r)^{2\lambda-1}}). \tag{15}$$

Consequently, for $n\to\infty$, $\frac{m}{n}\to 1$,

$$\frac{|D_n(\lambda)-D_{n-1}(\lambda)|}{d_n(2\lambda-1)}-\frac{|D_m(\lambda)-D_{m-1}(\lambda)|}{d_m(2\lambda-1)}$$

$$=\frac{n|D_n(\lambda)-D_{n-1}(\lambda)|-m|D_m(\lambda)-D_{m-1}(\lambda)|}{nd_n(2\lambda-1)}$$

$$-\frac{|D_m(\lambda)-D_{m-1}(\lambda)|}{d_m(2\lambda-1)}\frac{md_m(2\lambda-1)-nd_n(2\lambda-1)}{nd_n(2\lambda-1)}=o(1).$$

which is the required result.

Remark. Theorem B can be used to prove (1)[3] for $\frac{1}{2}<\lambda\leq\frac{3}{4}$, where the functions $f(z)$ are close-to-convex, but Milin's cannot.

References

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