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Difference Scheme for One Type of Nonlinear Parabolic Equation*

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We consider the following initial-boundary-value problem of nonlinear parabolic equations

$$\begin{cases} \dfrac{\partial}{\partial t}u_i(x,t) = \displaystyle\sum_{k=1}^{M}\frac{\partial}{\partial x}[a_{i,k}(x,t)\frac{\partial}{\partial x}u_k(x,t)] + F_i\Big(\frac{\partial u_1(x,t)}{\partial x},\cdots,\frac{\partial u_M(x,t)}{\partial x}, \\ \qquad\qquad u_1(x,t),\cdots,u_M(x,t)\Big),\ 0<x<1,\ 0<t<T,\ i=1,2,\cdots,M \quad (1) \\ u_i(x,0)=u_i^0(x),\ u_i(0,t)=\alpha_i(t),\ u_i(1,t)=\beta_i(t),\ i=1,2,\cdots,M, \quad (2) \end{cases}$$

Let h denote step size of space and τ denote step size of time. We use the following notation

$$U_{\bar{x}}^n = U_{\bar{x},j}^n = \frac{1}{h}(U_j^n - U_{j-1}^n),\quad U_x^n = U_{x,j}^n = \frac{1}{h}(U_{j+1}^n - U_j^n),\quad U_{\hat{x}}^n = U_{\hat{x},j}^n = \frac{1}{2h}(U_{j+1}^n - U_{j-1}^n),$$

$$U_{\bar{t}}^n = U_{\bar{t},j}^n = \frac{1}{\tau}(U_j^n - U_j^{n-1}),\quad (U,V) = h\sum_{j=1}^{J-1}U_j\cdot V_j,\quad (U,V] = h\sum_{j=1}^{J}U_jV_j,$$

$$\|U^n\|^2 = h\sum_{j=1}^{J-1}|U_j^n|^2,\quad \|U_{\bar{x}}^n\| = h\sum_{j=1}^{J}|U_{j,\bar{x}}^n|^2,\quad \|U^n\|_\infty = \sup_{0<j<J}|U_j^n|.$$

We give the difference scheme for the problem (1)-(2)

$$U_{i,\bar{t}}^{n+1} = \sum_{k=1}^{i-1}(a_{i,k}^{n+1/2}U_{k,\bar{x}}^{n+1})_x + \frac{1}{2}(a_{i,i}^{n+1/2}(U_{i,\bar{x}}^{n+1} + U_{i,\bar{x}}^n))_x + \sum_{k=i+1}^{M}(a_{i,k}^{n+1/2}U_{k,\bar{x}}^n)_x$$
$$+ F_i(U_{1,\hat{x}}^{n+1},\cdots,U_{i-1,\hat{x}}^{n+1},\ U_{i,\hat{x}}^n,\cdots,U_{M,\hat{x}}^n,\ U_1^{n+1},\cdots,U_{i-1}^{n+1},\ U_i^n,\cdots,U_M^n),$$
$$i=1,2,\cdots,M, \quad (3)$$

$$\begin{aligned} &U_{i,j}^0 = U_i^0(jh), && j=1,\ 2,\ \cdots J-1, \\ &U_{i,0}^n = \alpha_i(n\tau), \quad U_{i,J}^n = \beta_i(n\tau), && n=0,\ 1,\ \cdots,\ N, \end{aligned} \quad (4)$$

where $\qquad (a_{i,k})_j^n = a_{i,k}((j-\frac{1}{2})h,\ n\tau), \qquad 1<i,k<M.$

In the nonlinear term F_i of the scheme (3), unknown functions $u_{m,x}$ and u_m take the values in time $(n+1)\tau$ if $m<i$ and the values in time $n\tau$ if $m>i$. Approximation of the diffusion term is similar. But, because $|a_{i,i}|>|a_{i,k}|$ in general, we use average value of time $(n+1)\tau$ and time $n\tau$ for the thrm$\partial/\partial x(a_{i,i}\partial u_i/\partial x)$. This approximation can decrease error. Thus, in solving the difference equations

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in turn, we only need solve several linear algebraric equations of tridiagonal matrix. This scheme can save storage caqacity.

In order to prove convergence and stability of the scheme (3)-(4), we first consider the following difference problem

$$
\begin{cases}
z_{i,t}^{n+1} = \sum\limits_{k=1}^{i-1} (a_{i,k}^{n+1/2} z_{k,\bar{x}}^{n+1})_x + \frac{1}{2}(a_{i,i}^{n+1/2}(z_{i,\bar{x}}^{n+1} + z_{i,\bar{x}}^{n}))_x + \sum\limits_{k=i+1}^{M} (a_{i,k}^{n+1/2} z_{k,\bar{x}}^{n})_x + \\
\quad + [F_i(z_{1,\hat{x}}^{n+1} + y_{1,\hat{x}}^{n+1}, \cdots, z_{i-1,\hat{x}}^{n+1} + y_{i-1,\hat{x}}^{n+1}, z_{i,\hat{x}}^{n} + y_{i,\hat{x}}^{n}, \cdots, z_{M,\hat{x}}^{n} + y_{M,\hat{x}}^{n}, z_1^{n+1} + y_1^{n+1}, \cdots, \\
\quad z_{i-1}^{n+1} + y_{i-1}^{n+1}, z_i^n + v_i^n, \cdots, z_M^n + y_M^n) - F_i(y_{1,\hat{x}}^{n+1}, \cdots, y_{i-1,\hat{x}}^{n+1}, y_{i,\hat{x}}^{n}, \cdots, y_{M,\hat{x}}^{n}, y_1^{n+1}, \cdots, \\
\quad y_{i-1}^{n+1}, y_i^n, \cdots, y_M^n)] + R_i^n, \qquad n = 0, 1, \cdots, N, \ i = 1, 2, \cdots, M, \qquad (5) \\
z_{i,j}(x, 0) = \varphi_{i,j}, \qquad j = 1, 2, \cdots, J-1, \\
z_{i,0}^{n} = z_{i,J}^{n} = 0, \qquad (6)
\end{cases}
$$

Lemma. Assume that (i) $a_{i,k}(x,t)$ are symmetrical and uniform positive definite, *i. e.* $a_{i,k}(x,t) = a_{k,i}(x,t)$, $1 \le i, k \le M$ and exist constant $\sigma > 0$ such that there is

$$\sum_{i,k=1}^{M} a_{i,k}(x,t)\xi_i\xi_k \geq \sigma \sum_{i=1}^{M} \xi_i^2, \quad 0 \le x \le 1, \quad 0 \le t \le T, \quad \forall \ \text{real} \ (\xi_1, \xi_2, \cdots, \xi_M)$$

(ii) $|a_{i,k}(x,t)| \le A, \left|\frac{\partial}{\partial t} a_{i,k}(x,t)\right| \le A$, $1 \le i, \ k \le M$, where A is a positive constant,

(iii) the functions F_i satisfy Hölder condition with index 1 in R^{2M}, *i.e.*

$|F_i(r_1, r_2, \cdots, r_{2M}) - F_i(s_1, s_2, \cdots, s_{2M})| \le C_F \cdot [|r_1 - s_1|^2 + |r_2 - s_2|^2 + \cdots + |r_{2M} - s_{2M}|^2]^{1/2}$, $1 \le i \le M$, where C_F is a positive constant. Then there is estimate for the problem (5)-(6)

$$\sum_{i=1}^{M} \| z_{i,\bar{x}}^{n} \|^2 \le C_1 \left[\sum_{i=1}^{M} \| \varphi_{i,k} \|^2 + \max_{0 \le l \le [T/\tau]} \sum_{i=1}^{M} \| R_i^l \|^2 \right] e^{C_2 T}, \qquad (7)$$

where C_1 and C_2 are positive constants.

Proof. Rewrite (5) as the following form

$$
\begin{aligned}
z_{i,t}^{n+1} = {} & \frac{1}{2} \sum_{k=1}^{M} \left[a_{i,k}^{n+1/2}(z_{k,\bar{x}}^{n+1} + z_{k,\bar{x}}^{n})\right]_x + \frac{1}{2} \sum_{k=1}^{i-1} [a_{i,k}^{n+1/2}(z_{k,\bar{x}}^{n+1} - z_{k,\bar{x}}^{n})]_x - \frac{1}{2} \sum_{k=i+1}^{M} [a_{i,k}^{n+1/2}(z_{k,\bar{x}}^{n+1} \\
& - z_{k,\bar{x}}^{n})]_x + [F_i(z_{1,\hat{x}}^{n+1} + y_{1,\hat{x}}^{n+1}, \cdots, z_{i-1,\hat{x}}^{n+1} + y_{i-1,\hat{x}}^{n+1}, z_{i,\hat{x}}^{n} + y_{i,\hat{x}}^{n}, \cdots z_{M,\hat{x}}^{n} + y_{M,\hat{x}}^{n}, z_1^{n+1} + y_1^{n+1}, \\
& \cdots, z_{i-1}^{n+1} + y_{i-1}^{n+1}, z_i^n + y_i^n, \cdots, z_m^n + y_M^n) - F_i(y_{1,\hat{x}}^{n+1}, \cdots, y_{i-1,\hat{x}}^{n+1}, y_{i,\hat{x}}^{n}, \cdots, y_{M,\hat{x}}^{n}, y_1^{n+1}, \cdots, \\
& y_{i-1}^{n+1}, y_i^n, \cdots, y_M^n)] + R_i^n, \qquad n = 0, 1, \cdots; N, \quad i = 1, 2, \cdots, M. \qquad (8)
\end{aligned}
$$

Multiplylng i-th equation of equations (8) by $z_{i,\bar{t}}^{n+1}$ and taking the inner product, we obtain

$$
\begin{aligned}
(z_{i,\bar{t}}^{n+1}, z_{i,\bar{t}}^{n+1}) = {} & \frac{1}{2}\left(z_{i,\bar{t}}^{n+1}, \sum_{k=1}^{M}[a_{i,k}^{n+1/2}(z_{k,\bar{x}}^{n+1} + z_{k,\bar{x}}^{n})]_x\right) + \frac{1}{2}\left(z_{i,\bar{t}}^{n+1}, \sum_{k=1}^{i-1}[a_{k,x}^{n+1/2}(z_{k,\bar{x}}^{n+1} - z_{k,\bar{x}}^{n})]_x\right) \\
& - \frac{1}{2}\left(z_{i,\bar{t}}^{n+1}, \sum_{k=i+1}^{M}[a_{i,k}^{n+1/2}(z_{k,\bar{x}}^{n+1} - z_{k,\bar{x}}^{n})]_x\right) + (z_{i,\bar{t}}^{n+1}, F_i(z_{1,\hat{x}}^{n+1} + y_{1,\hat{x}}^{n+1} \cdots, z_M^n + y_M^n) -
\end{aligned}
$$

$$- F_i(y^{n+1}_{1,\hat{x}}, \cdots, y^n_M)) + (z^{n+1}_{i,\bar{t}}, R^n_i).$$

Summing up the last formula for i from 1 to M and using Green's formula of difference operetor, we have

$$\sum_{i=1}^{M}\| z^{n+1}_{i,\bar{t}}\|^2 = -\frac{1}{2}\sum_{i,k=1}^{M}(z^{n+1}_{i,\bar{t},\bar{x}}, a^{n+1/2}_{i,k}(z^{n+1}_{k,\bar{x}} + z^n_{k,\bar{x}})] - \frac{1}{2}\sum_{i=1}^{M}\sum_{k=1}^{i-1}(z^{n+1}_{i,\bar{t},\bar{x}}, \tau a^{n+1/2}_{i,k} z^{n+1}_{k,\bar{x},\bar{t}}) +$$

$$+\frac{1}{2}\sum_{i=1}^{M}\sum_{k=i+1}^{M}(z^{n+1}_{i,\bar{t},\bar{x}}, \tau a^{n+1/2}_{i,k} z^{n+1}_{k,\bar{x},\bar{t}}] + \sum_{i=1}^{M}(z^{n+1}_{i,\bar{t}}, F_i(z^{n+1}_{1,\hat{x}} + y^{n+1}_{1,\hat{x}}, \cdots, z^n_M + y^n_M) -$$

$$- F_i(y^{n+1}_{1,\hat{x}}, \cdots, y^n_M)) + \sum_{i=1}^{M}(z^{n+1}_{i,\bar{t}}, R^n_i), \tag{9}$$

Because $a_{i,k}$ satisfy condition (i), we can obtain

$$\sum_{i,k=1}^{M}(z^{n+1}_{i,\bar{x}}, a^{n+1/2}_{i,k} z^n_{k,\bar{x}}] = \sum_{i,k=1}^{M}(z^n_{i,\bar{x}}, a^{n+1/2}_{i,k} z^{n+1}_{k,\bar{x}}]$$

and $\displaystyle\sum_{i=1}^{M}\sum_{k=1}^{i-1}(z^{n+1}_{i,\bar{t},\bar{x}}, a^{n+1/2}_{i,k} z^{n+1}_{k,\bar{x},\bar{t}}] = \sum_{i=1}^{M}\sum_{k=i+1}^{M}(z^{n+1}_{i,\bar{t},\bar{x}}, a^{n+1/2}_{i,k} z^{n+1}_{k,\bar{x},\bar{t}}]$.

Thus, (9) yield

$$\sum_{i=1}^{M}\| z^{n+1}_{i,\bar{t}}\|^2 + \frac{1}{2\tau}\sum_{i,k=1}^{M}(z^{n+1}_{i,\bar{x}}, a^{n+1/2}_{i,k} z^{n+1}_{k,\bar{x}}] = \frac{1}{2\tau}\sum_{i,k=1}^{M}(z^n_{i,\bar{x}}, a^{n+1/2}_{i,k} z^n_{k,\bar{x}}]$$

$$+ \sum_{i=1}^{M}(z^{n+1}_{i,\bar{t}}, F_i(z^{n+1}_{1,\hat{x}} + y^{n+1}_{1,\hat{x}}, \cdots, z^n_M + y^n_M) - F_i(y^{n+1}_{1,\hat{x}}, \cdots, y^n_M)) + \sum_{i=1}^{M}(z^{n+1}_{i,\bar{t}}, R^n_i), \tag{10}$$

Now we estimate terms in the formula (10) as follows

$$\left|\frac{1}{2\tau}\sum_{i,k=1}^{M}(z^n_{i,\bar{x}}, a^{n+1/2}_{i,k} z^n_{k,\bar{x}}]\right|$$

$$\leqslant \frac{A}{2\tau}\sum_{i,k=1}^{M}\frac{1}{2}(\| z^n_{i,\bar{x}}\|^2 + \| z^n_{k,\bar{x}}\|^2) \leqslant \frac{AM}{2\tau}\sum_{i=1}^{M}\| z^n_{i,\bar{x}}\|^2,$$

$$\left|\sum_{i=1}^{M}(z^{n+1}_{i,\bar{t}}, F_i(z^{n+1}_{1,\hat{x}} + y^{n+1}_{1,\hat{x}}, \cdots, z^n_M + y^n_M) - F_i(y^{n+1}_{1,\hat{x}}, \cdots, y^n_M))\right|$$

$$\leqslant \frac{1}{2}\sum_{i=1}^{M}\| z^{n+1}_{i,\bar{t}}\|^2 + \frac{1}{2}\sum_{i=1}^{M}\| F_i(z^{n+1}_{1,\hat{x}} + y^{n+1}_{1,\hat{x}}, \cdots, z^n_M + y^n_M) - F_i(y^{n+1}_{1,\hat{x}}, \cdots, y^n_M)\|^2$$

$$\leqslant \frac{1}{2}\sum_{i=1}^{M}\| z^{n+1}_{i,\bar{t}}\|^2 + \frac{1}{2}\sum_{i=1}^{M} C_F\Big[\sum_{k=1}^{i-1}(\| z^{n+1}_{k,\hat{x}}\|^2 + \| z^{n+1}_k\|^2) + \sum_{k=i}^{M}(\| z^n_{k,\hat{x}}\|^2 + \| z^n_k\|^2)\Big]$$

$$\leqslant \frac{1}{2}\sum_{i=1}^{M}\| z^{n+1}_{i,\bar{t}}\|^2 + \frac{1}{2}MC_F\sum_{i=1}^{M}(\| z^{n+1}_{i,\hat{x}}\|^2 + \| z^{n+1}_n\|^2 + \| z^n_{i,\hat{x}}\|^2 + \| z^n_i\|^2),$$

$$\left|\sum_{i=1}^{M}(z^{n+1}_{i,\bar{t}}, R^n_i)\right| \leqslant \frac{1}{2}\sum_{i=1}^{M}(\| z^{n+1}_{i,\bar{t}}\|^2 + \| R^n_i\|^2) \quad .$$

Using the preceding deduction, from (10) we obtain

$$\left|\sum_{i=1}^{M}\| z^{n+1}_{i,\bar{t}}\|^2 + \frac{1}{2\tau}\sum_{i,k=1}^{M}(z^{n+1}_{i,\bar{x}}, a^{n+1/2}_{i,k} z^{n+1}_{k,\bar{x}}]\right| \leqslant \frac{1}{2\tau}AM\sum_{i=1}^{M}\| z^n_{i,\bar{x}}\|^2 + \frac{1}{2}\sum_{i=1}^{M}\| z^{n+1}_{i,\bar{t}}\|^2 + \frac{1}{2}$$

$$+\frac{1}{2}MC_F\sum_{i=1}^{M}(\| z^{n+1}_{i,\hat{x}}\|^2 + \| z^{n+1}_i\|^2 + \| z^n_{i,\hat{x}}\|^2 + \| z^n_i\|^2) + \frac{1}{2}\sum_{i=1}^{M}\| z^{n+1}_{i,\bar{t}}\|^2 + \frac{1}{2}\sum_{i=1}^{M}\| R^n_i\|^2 \quad . \tag{11}$$

In view of the condition of uniform positive definite and $\|z^n_{i,x}\| \leqslant \|z^n_{i,\bar{x}}\|$, we have

$$\sum_{i=1}^{M}\|z^{n+1}_{i,\bar{x}}\|^2 \leqslant \frac{1}{\sigma}AM\sum_{i=1}^{M}\|z^n_{i,\bar{x}}\|^2 + \frac{1}{\sigma}\tau\sum_{i=1}^{M}[MC_F(\|z^{n+1}_{i,\bar{x}}\|^2 + \|z^{n+1}_i\|^2 + \|z^n_{i,\bar{x}}\|^2 + \|z^n_i\|^2) + \|R^n_i\|^2] .$$

Use the inequality $\|z^n_i\| \leqslant C_3\|z^n_{i,\bar{x}}\|$, where C_3 is a positive constant, we obtain

$$\sum_{i=1}^{M}\|z^{n+1}_{i,\bar{x}}\|^2 \leqslant \frac{1}{\sigma}AM\sum_{i=1}^{M}\|z^0_{i,\bar{x}}\|^2 + \frac{1}{\sigma}2MC_F(C_3+1)\tau\sum_{l=0}^{n+1}\sum_{i=1}^{M}\|z^l_{i,\bar{x}}\|^2 + \frac{\tau}{\sigma}\sum_{l=0}^{n}\sum_{i=1}^{M}\|R^l_i\|^2$$

$$\leqslant \frac{1}{\sigma}AM\sum_{i=1}^{M}\|\varphi_{i,\bar{x}}\|^2 + \frac{1}{\sigma}2MC_F(C_3+1)\tau\sum_{l=0}^{n+1}\sum_{i=1}^{M}\|z^l_{i,\bar{x}}\|^2 + \frac{1}{\sigma}\tau\max_{0\leqslant l\leqslant[T/\tau]}\sum_{i=1}^{M}\|R^l_i\|^2, \qquad (12)$$

In view of Gronwall's inequality of difference operetor we have

$$\sum_{i=1}^{M}\|z^n_{i,\bar{x}}\|^2 \leqslant C_1\Big[\sum_{i=1}^{M}\|\varphi_{i,\bar{x}}\|^2 + \max_{0\leqslant l\leqslant[T/\tau]}\sum_{i=1}^{M}\|R^l_i\|^2\Big]e^{C_2T} .$$

Theorem 1. Suppose the conditions of Lemma are satisfied. Assume that(i) the solution of the problem (1)-(2) possesses bounded partial derivative of fourth order for x and bounded partial derivative of second order for t, $a_{i,k}(x,t)$ possesses bounded partial derivative of third order for x, $1\leqslant i,\ k\leqslant M$. Then the solution of the difference scheme (3)-(4) converges to the solution of the problem (1)-(2) with order $O(\tau+h^2)$ by L_∞ norm.

Proof. Making Taylor's expansion, it follows from (3)

$$u^{n+1}_{i,\bar{t}} = \sum_{k=1}^{i-1}(a^{n+1/2}_{k,x}u^{n+1}_{k,\bar{x}})_x + \frac{1}{2}(a^{n+1/2}_{i,i}(u^{n+1}_{i,\bar{x}} + u^n_{i,\bar{x}}))_x + \sum_{k=i+1}^{M}(a^{n+1/2}_{i,k}u^n_{k,\bar{x}})_x$$
$$+ F_i(u^{n+1}_{1,\hat{x}}, \cdots, u^{n+1}_{i-1,\hat{x}}, u^n_{i,\hat{x}}, \cdots, u^n_{M,\hat{x}}, u^{n+1}_1, \cdots, u^{n+1}_{i-1}, u^n_i, \cdots, u^n_M) + R^n_i, \qquad (13)$$

where $|R^n_i| \leqslant \eta(\tau+h^2)$, η is a positive constant. Let $z_i(x,t) = u_i(x,t) - U_i(x,t)$. We can establish error equations

$$z^{n+1}_{i,\bar{t}} = \sum_{k=1}^{i-1}(a^{n+1/2}_{i,k}z^{n+1}_{k,\bar{x}})_x + \frac{1}{2}(a^{n+1/2}_{ii}(z^{n+1}_{i,\bar{x}} + z^n_{i,\bar{x}}))_x + \sum_{k=i+1}^{M}(a^{n+1/2}_{i,k}z^n_{k,\bar{x}})_x + F_i(z^{n+1}_{1,\hat{x}} +$$
$$+ U^{n+1}_{1,\hat{x}}, \cdots, z^{n+1}_{i-1,\hat{x}} + U^{n+1}_{i-1,\hat{x}}, z^n_{i,\hat{x}} + U^n_{i,\hat{x}}, \cdots, z^n_{M,\hat{x}} + U^n_{M,\hat{x}}, z^{n+1}_1 + U^{n+1}_1, \cdots, z^{n+1}_{i-1} +$$
$$+ U^{n+1}_{i-1}, z^n_i + U^n_i, \cdots, z^n_M + U^n_M) - F_i(U^{n+1}_{1,\hat{x}}, \cdots, U^{n+1}_{i-1,\hat{x}}, U^n_{i,\hat{x}}, \cdots, U^n_{M,\hat{x}}, U^{n+1}_1, \cdots, U^{n+1}_{i-1},$$
$$U^n_i, \cdots, U^n_M) + R^n_i , \qquad i = 1, 2, \cdots, M. \qquad (14)$$

$$z^0_{i,j} = 0,\ j = 1,2,\cdots,J-1,\ z^n_{i,0} = 0,\ z^n_{i,J} = 0,\ n = 0,1,\cdots,N,\ i = 1,2,\cdots,M, \qquad (15)$$

It follows from the Lemma

$$\sum_{i=1}^{M}\|z^n_{i,\bar{x}}\|^2 \leqslant C_1\max_{0\leqslant l\leqslant[T/\tau]}\sum_{i=1}^{M}\|R^l_i\|^2e^{C_2T} \leqslant C_1M\eta(\tau+h^2)e^{C_2T} .$$

In view of the inequality

$\sum_{i=1}^{M}\|z^n_i\|^2_{L_\infty} \leqslant \text{const}\sum_{i=1}^{M}\|z^n_{i,\bar{x}}\|^2$, we obtain convergence by L_∞ norm.

Theorem 2. Suppose the conditions of the Lemma are satisfy, then the solution of the difference problem (3)-(4) is stable for initial value.

proof. Suppose that there are the solutions $(U_i)_j^n$ and $(V_i)_j^n$, which satisfy the difference equation (3) and homogeneous boundary condition. But, their initial values are different: $(U_i)_j^0=u_{i,j}^0$, $(V_i)_j^0=v_{i,j}^0$. Similar to the proof of the Theorem 1, we can obtain

$$\sum_{i=1}^{M}\|U_{i,j}^n-V_{i,j}^n\|_{L_\infty}^2<\text{const}\sum_{i=1}^{M}\|(u_{i,j}^0-v_{i,j}^0)_x\|^2,$$

i.e. the difference scheme is stable.

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