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Bivariate Interpolating Polynomials and Remainders on Triangular Region*

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The Interpolation of degree $2n-1$ for $f(x,y)$ on the triangular region with the vertices at $(0,0),(0,1),(1,1)$ can be expressed by

$$P_{1,2n-1}(x,y)=\sum_{\beta=0}^{n-1}\Big[\sum_{a=0}^{\beta}f_{x^a y^{2\beta-a}}(0,0)P_{2\beta+1}^{a,2\beta-a}(x,y)+\sum_{a=0}^{\beta}f_{x^{2\beta-a}y^a}(1,1)Q_{2\beta+1}^{2\beta-a,a}(x,y)+\sum_{a=0}^{2\beta}f_{x^a y^{2\beta-a}}(0,1)R_{2\beta+1}^{a,2\beta-a}(x,y)\Big].$$

It satisfies the following interpolating conditions:

$$\left.\frac{\partial^{2\beta}P_{1,2n-1}(x,y)}{\partial x^a\partial y^{2\beta-a}}\right|_{(0,0)}=f_{x^a y^{2\beta-a}}(0,0),\qquad 0\leqslant a\leqslant\beta,\ \beta=0,1,\cdots,n-1$$

$$\left.\frac{\partial^{2\beta}P_{1,2n-1}(x,y)}{\partial x^a\partial y^{2\beta-a}}\right|_{(0,1)}=f_{x^a y^{2\beta-a}}(0,1),\qquad 0\leqslant a\leqslant 2\beta,\ \beta=0,1,\cdots,n-1$$

$$\left.\frac{\partial^{2\beta}P_{1,2n-1}(x,y)}{\partial x^{2\beta-a}\partial y^a}\right|_{(1,1)}=f_{x^{2\beta-a}y^a}(1,1),\qquad 0\leqslant a\leqslant\beta,\ \beta=0,1,\cdots,n-1$$

The functions $P's, Q's$ and $R's$ can be determined by the following rules: $P_1^{0,0}(x,y)=1-y$, $Q_1^{0,0}(x,y)=x$, $R_1^{0,0}(x,y)=y-x$; $P_{2\beta+1}^{0,2\beta}(x,y)=f_{2\beta+1}(y)$, $R_{2\beta+1}^{0,2\beta}(x,y)=g_{2\beta+1}(y)$, $(\beta\geqslant1)$; $Q_{2\beta+1}^{2\beta,0}(x,y)=g_{2\beta+1}(x)$, $R_{2\beta+1}^{2\beta,0}(x,y)=f_{2\beta+1}(x)$, $(\beta\geqslant1)$; $g_{2\beta+1}(t)=\int_0^1 du\int_0^u g_{2\beta-1}(v)dv-t\int_0^1 du\int_0^u g_{2\beta-1}(v)dv$, $\beta\geqslant1$, $g_1(t)=t$; $f_{2\beta+1}(t)=g_{2\beta+1}(1-t)$, $\beta=0,1,2,\cdots$; $P_{2\beta+1}^{a,2\beta-a}(x,y)=\int_0^x du\int_1^y P_{2\beta-1}^{a-1,2\beta-a-1}(u,v)dv$, $Q_{2\beta+1}^{2\beta-a,a}(x,y)=\int_0^x du\int_1^y Q_{2\beta-1}^{2\beta-a-1,a-1}(u,v)dv$, $1\leqslant a\leqslant\beta$; $R_{2\beta+1}^{a,2\beta-a}(x,y)=\int_0^x du\int_1^y R_{2\beta-1}^{a-1,2\beta-a-1}(u,v)dv$, $1\leqslant a\leqslant 2\beta-1$.

Theorem If $f(x,y)$ is continuous up to the $2n$ th partial derivatives, then we have for $n\geqslant2$.

$$\begin{aligned}f(x,y)-P_{1,2n-1}(x,y)&=\sum_{a=0}^{n-1}f_{x^a y^{2n-a}}(0,\eta_a)[P_{2n+1}^{a,2n-a}(x,y)+R_{2n+1}^{a,2n-a}(x,y)]\\&+\sum_{a=0}^{n-1}f_{x^{2n-a}y^a}(\xi_a,1)[R_{2n+1}^{2n-a,a}(x,y)+Q_{2n+1}^{2n-a,a}(x,y)]+f_{x^n y^n}(\xi_n,\eta_n)[P_{2n+1}^{n,n}(x,y)+Q_{2n+1}^{n,n}(x,y)\\&+R_{2n+1}^{n,n}(x,y)]+\frac{1}{(2n)!}\sum_{\substack{a=n-2\\a\neq n}}^{n+2}(-1)^a\binom{2n}{a}[f_{x^a y^{2n-a}}(\xi_a^*,\eta_a^*)-f_{x^a y^{2n-a}}(\lambda_a,\mu_a)]x^a(1-y)^{2n-a},\end{aligned}$$

$0<\xi_a<1$, $0<\eta_a<1$, $(a=0,1,\cdots,n)$; $0<\xi_a^*<1$, $0<\eta_a^*<1$, $0<\lambda_a<1$, $0<\mu_a<1$, $(a=n-2,n-1,n+1,n+2)$

It is easy to transform the interpolation $P_{1,2n-1}(x,y)$ of $f(x,y)$ and the remainder on an arbitrarily triangular region.

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