# 一类非线性退缩抛物方程的混合问题*

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## §1　引　　言

设$Q_T=\{(x,t);0<x<\infty,\ 0<t\leq T\}$，我们将在$Q_T$上讨论如下形式的混合Cauchy Dirichlet问题

$$u_t=(a(u)u_x)_x+b(u)u_x, \tag{1.1}$$

$$u(x,0)=u_0(x),\quad 0\leq x<\infty, \tag{1.2}$$

$$u(0,t)=\psi(t),\quad 0\leq t\leq T, \tag{1.3}$$

式中$a(0)=0$，当$s>0$时，$a(s)>0$。

文〔1〕证明了问题（1.1）－（1.3）存在非负连续解，并在$b^2(s)=O(a(s))$，$s\to O^+$的条件下证明了解的唯一性．本文给出了问题（1.1）－（1.3）存在非负连续且具有界变差解的充分条件，同时也证明了这种连续且具有界变差解是唯一的（不需加$b^2(s)=O(a(s))$，$s\to O^+$的条件）．用本文中所使用的方法同样可以讨论相应的初值问题和第一边值问题，从而得到类似的结果．

记$A(u)=\int_0^u a(\tau)d\tau$，$B(u)=\int_0^u b(\tau)d\tau$.

**定义**　函数$u(x,t)\in C(\overline{Q}_T)$称为问题（1.1）—（1.3）的广义解，如果

1°　$u(x,t)\in BV(Q_T)$．2°　$u(x,t)$在$\overline{Q}_T$上非负且有界．

3°　对任何$\varphi(x,t)\in C^{2,1}_{x,t}(\overline{Q}_T)$，$\varphi(0,t)=\varphi(x,T)=0$，当$x$充分大时$\varphi(x,t)=0$，$u(x,t)$满足

$$\iint_{Q_T}[u\varphi_t+A(u)\varphi_{xx}-B(u)\varphi_x]dxdt=-\int_0^\infty u_0(x)\varphi(x,0)dx-\int_0^T A(\psi(t))\varphi_x(0,t)dt. \tag{1.4}$$

## §2　广义解的存在性

在本节和下一节的讨论中，我们作如下假设：A　$a(s)$、$b(s)$适当光滑．　B　$u_0(x)$在$[0,\infty)$上非负有界，$u_0'(x)$在$[0,\infty)$上Lipschitz连续，$\psi(t)\in C^2[0,T]$，$\psi(t)>0$，$0\leq t\leq T$，$u_0(0)=\psi(0)$．

首先指出，在上述假设条件下，存在函数序列$\{u_{0n}(x)\}$和$\{\psi_n(t)\}$及正常数序列$\{\varepsilon_n\}$，它们满足

1°　$u_n(x)$　$C^\infty[0,\infty)$，$\psi_n(t)\in C^\infty[0,T]$，$n=1,2,\cdots$.

2°　$\varepsilon_n\leq u_{0n}(x)\leq M$，当$x<n$时；$u_{0n}(x)=M$，当$x\geq n$时；$\varepsilon_n\leq\psi_n(t)\leq M$，$t\in[0,T]$；$n=1,2,\cdots$.

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3° $u_{0n}(x)>0$，$u_{0n+1}(x)\le u_{0n}(x)$，$x\in[0,\infty)$，$n=1,2\cdots$.

$\psi_n(t)>0$，$\psi_{n+1}(t)\le\psi_n(t)$，$t\in[0,T]$，$n=1,2,\cdots$.

4° $|u'_{0n}(x)|$、$|u''_{0n}(x)|$在〔0，∞）上对$n$一致有界，$|\psi_n(t)|$、$|\psi_n(t)|$、$|\psi''_n(t)|$在〔0，$T$〕上对$n$一致有界.

5° 当$n\to\infty$时，在〔0，∞）的任一紧致集上一致地有$u_{0n}\to u_0$，在〔0，$T$〕上一致地有$\psi_n\to\psi$.

6° $\psi_n(0)=u_{0n}(0)$，$u_{0n}^{(i)}(0)=u_{0n}^{(i)}(n)=0$，$i=1,2,3,4.\psi'_n(0)=\psi''_n(0)=0,n=1,2,3,\cdots$.

7° $\varepsilon_n$单调递减趋于零.

为证明广义解的存在性，我们考虑如下的边值问题

$$u_t=(a(u)u_x)_x+b(u)u_x \tag{2.1}$$

$$u(x,0)=u_{0n}(x) \tag{2.2}$$

$$u(0,t)=\psi_n(t),\quad u(n,t)=M \tag{2.3}$$

由文〔1〕、〔2〕可知，问题（2.1）—（2.3）存在唯一解$u_n\in C^{2;1}_{x,t}(\overline{Q}_T)\cap C^{3;2}_{x,t}(Q_T)$.

**引理1**（极值原理） 设$Q^n_T=\{(x,t);\ 0<x<n,\ 0<t\le T\}$，$\Gamma=\overline{Q}^n_T\setminus Q^n_T$. 则问题（2.1）—（2.3）的解$u_n$在边界$\Gamma$上取其最大值与最小值.

**证明** 只证最大值这一情形，最小值同理可证. 令$m_1=\max\limits_{(x,t)\in\overline{Q}^n_T}u_n(x,t)$，$m_2=\max\limits_{(x,t)\in\Gamma}u_n(x,t)$. 假设结论不真，即$m_1>m_2$，则存在$(x_1,t_1)\in Q^n_T$，使得$u_n(x_1,t_1)=m_1$. 作辅助函数$v(x,t)=u_n(x,t)+\dfrac{m_1-m_2}{4T}(t_1-t)$. 在$\Gamma$上有$v(x,t)\le m_2+\dfrac{m_1-m_2}{4}=\dfrac{m_1}{4}+\dfrac{3}{4}m_2<m_1$，而$v(x_1,t_1)=u_n(x_1,t_1)=m_1$，可见$v(x,t)$亦不在$\Gamma$上取其最大值. 设$v(x,t)$在点$(x_2,t_2)\in Q^n_T$处取最大值，则在$(x_2,t_2)$点有$v_x=0$，$v_{xx}\le0$，$v_t\ge0$. 从而，在点$(x_2,t_2)$处有$v_t-(a(u_n)v_x)_x-b(u_n)v_x=v_t-a(u_n)v_{xx}\ge0$. 另一方面，在点$(x_2,t_2)$处$v_t-(a(u_n)v_x)_x-b(u_n)v_x=\dfrac{m_1-m_2}{4T}-(a(u_n)(u_n)_x)_x-b(u_n)(u_n)_x=-\dfrac{m_1-m_2}{4T}<0$. 矛盾. 故$u_n(x,t)$在$\Gamma$上取得最大值.

根据引理1知，问题（2.1）—（2.3）的解$u_n(x,t)$满足$0<u_n(x,t)\le M$.

**命题** 设存在$\eta>0$，使得当$|u|\le M$（$M$是（2.4）中的常数）时

$$a(u)-\eta a_u^2(u)\ge0, \tag{*}$$

则问题（2.1）—（2.3）的解$u_n$满足

$$\int_0^n|\text{grad}\,u_n|w_\lambda(x)\mathrm{d}x\le M_1.$$

这里，$\text{grad}\,u=(u_x,u_t)$，$w_\lambda(x)=\exp(-\lambda\sqrt{1+x^2})$，$M_1$为与$n$无关的常数.

**证明** 令$V(x,t)=u(x,t)-\psi(x,t)$，其中$\psi(x,t)>0$，$\psi(x,t)\in C^3(\overline{Q}_T)$，$\psi(0,t)=\psi_n(t)$，$\psi(n,t)=M$. 则问题（2.1）—（2.3）可改写为

$$V_t=(a(V+\psi)V_x)_x+(f(V,x,t))_x+g(V,x,t) \tag{2.5}$$

$$V(x,0)=V_{0n}(x) \tag{2.6}$$

$$V(0,t)=V(n,t)=0 \tag{2.7}$$

这里，$f(V,x,t)=\int_0^V(b(\tau+\psi)+a_u(\tau+\psi)\psi_x)\mathrm{d}\tau$，$g(V,x,t)=a_u(V+\psi)\psi_x^2+b\psi_x-\psi_t-f_x-a(V+\psi)\psi_{xx}$．$V_{0n}(x)=u_{0n}(x)-\psi(x,0)$．

为证明本命题，先给出一个引理．

引理 2　设$\psi(t)>0$，$0\leq t\leq T$．则问题（2.5）—（2.7）的解$V_n$满足

$$\left(\left|\frac{\partial V_n}{\partial x}\right|w_\lambda\right)\Big|_{x=0,n}\leq C_1+C_2\int_0^n|\mathrm{grad}V_n|w_\lambda(x)\mathrm{d}x.$$

其中，$C_1$和$C_2$是与$n$无关的常数．

证明　记$f(x,t)=\frac{\partial}{\partial x}\left(a(V_n+\psi)\cdot w_\lambda\cdot\frac{\partial V_n}{\partial x}\right)$．

$$f^+(x,t)=\begin{cases}f(x,t) & f(x,t)>0,\\ 0 & f(x,t)\leq 0,\end{cases}\qquad f^-(x,t)=\begin{cases}-f(x,t) & f(x,t)<0,\\ 0 & f(x,t)\geq 0.\end{cases}$$

对任何固定的$t\in[0,T]$，考虑问题

$$\frac{\partial}{\partial x}\left(a(V_n+\psi)w_\lambda\frac{\partial V_{1\delta}}{\partial x}\right)=f^-,\qquad V_1|_{x=0,n}=0.$$

由问题

$$\frac{\partial}{\partial x}\left(a(V_n+\psi)w_\lambda\frac{\partial V_{1\delta}}{\partial x}\right)-\delta V_{1\delta}=f^-(x,t)\quad(\delta>0).\quad V_{1\delta}|_{x=0,n}=0$$

的解$V_{1\delta}$的非正性，可知$V_1$非正，进而知$\frac{\partial V_1}{\partial x}\Big|_{x=0}\leq 0$，$\frac{\partial V_1}{\partial x}\Big|_{x=n}\geq 0$．再注意到$V_n|_{x=0,n}=0$，便得到$\int_0^n f^-\mathrm{d}x=\left(a(V_n+\psi)w_\lambda\frac{\partial V_1}{\partial x}\right)\Big|_{x=0}=a(M)e^{-\lambda\sqrt{1+n^2}}\left|\frac{\partial V_1}{\partial x}\right|_{x=n}+a(\psi_n)e^{-\lambda}\left|\frac{\partial V_1}{\partial x}\right|_{x=0}$．

同理，再考虑问题

$$\frac{\partial}{\partial x}\left(a(V_n+\psi)w_\lambda\frac{\partial V}{\partial x}\right)=f^-,\ V_2|_{0,n}=0.\ \text{可得}$$

$$\int_0^n f^-\mathrm{d}x=a(M)e^{-\lambda\sqrt{1+n^2}}\left|\frac{\partial V_2}{\partial x}\right|_{x=n}+a(\psi_n)e^{-\lambda}\left|\frac{\partial V_2}{\partial x}\right|_{x=0}.$$

因为函数$V_1—V_2$是问题

$$\frac{\partial}{\partial x}\left(a(V_n+\psi)w_\lambda\frac{\partial V}{\partial x}\right)=f(x,t).\qquad V|_{x=0,n}=0$$

的解．由解的唯一性知$V_1-V_2=V_n$．从而

$$\left(\left|\frac{\partial V_n}{\partial x}\right|w_\lambda\right)\Big|_{x=0}+\left(\left|\frac{\partial V_n}{\partial x}\right|w_\lambda\right)\Big|_{x=n}\leq\left(\left|\frac{\partial V_1}{\partial x}\right|w_\lambda\right)\Big|_{x=0}+\left(\left|\frac{\partial V_1}{\partial x}\right|w_\lambda\right)\Big|_{x=n}+$$

$$\left(\left|\frac{\partial V_2}{\partial x}\right|w_\lambda\right)\Big|_{x=0}+\left(\left|\frac{\partial V_2}{\partial x}\right|w_\lambda\right)\Big|_{x=n}\leq K\Big[\left(a\left|\frac{\partial V_1}{\partial x}\right|w_\lambda\right)\Big|_{x=0}+\left(a\left|\frac{\partial V_1}{\partial x}\right|w_\lambda\right)\Big|_{x=n}+$$

$$\left(a\left|\frac{\partial V_2}{\partial x}\right|w_\lambda\right)\Big|_{x=0}+\left(a\left|\frac{\partial V_2}{\partial x}\right|w_\lambda\right)\Big|_{x=n}=K\left[\int_0^n f^-\mathrm{d}x+\int_0^n f^+\mathrm{d}x\right]=K\int_0^n|f|\mathrm{d}x=$$

$$K\int_0^n\left|\frac{\partial}{\partial x}\left(a\frac{\partial V_n}{\partial x}w_\lambda\right)\right|\mathrm{d}x\leq K\int_0^n\left|\frac{\partial}{\partial x}\left(a\frac{\partial V_n}{\partial x}\right)\right||w_\lambda|\mathrm{d}x+K\int_0^n a\left|\frac{\partial V_n}{\partial x}\right|\cdot|w_\lambda'|\mathrm{d}x.$$

再利用方程（2.5）并注意到$|w_\lambda'(x)|\leq\lambda w_\lambda(x)$，便证明了引理 2．（$K$是与$n$无关的正常数）．

现在，仍回到我们要证明的命题．显然，为证明命题的结论，只须证明

$$\int_0^n |\operatorname{grad} V_n| w_\lambda(x)\mathrm{d}x \le M_2.$$

设 $\operatorname{sgn}_\eta \tau = \begin{cases} 1 & (\tau > \eta) \\ \dfrac{\tau}{\eta} & (|\tau| \le \eta) \\ -1 & (\tau < -\eta) \end{cases} \qquad (\eta > 0).$

对 $\xi=(\xi_1, \xi_2)$，设 $|\xi| = (\xi_1^2+\xi_2^2)^{1/2}$，$I_\eta(\xi) = \int_0^{|\xi|} \operatorname{sgn}_\eta \tau \mathrm{d}\tau$.

将（2.5）对 $x$ 微分后乘以 $V_x \dfrac{\operatorname{sgn}_\eta |\operatorname{grad}V|}{|\operatorname{grad}V|} w_\lambda(x)$．再将（2.5）对 $t$ 微分后乘以 $V_t \dfrac{\operatorname{sgn}_\eta |\operatorname{grad}V|}{|\operatorname{grad}V|} w_\lambda(x)$，然后两式相加，在（0，$n$）上积分，得

$$\frac{d}{dt}\int_0^n I_\eta(\operatorname{grad}V) w_\lambda(x)\mathrm{d}x - \int_0^n \frac{\partial}{\partial x}(a_u V_x^2 + a_u \psi_x V_x + a V_{xx}) V_x \frac{\operatorname{sgn}_\eta |\operatorname{grad}V|}{|\operatorname{grad}V|} w_\lambda(x)\mathrm{d}x -$$

$$\int_0^n \frac{\partial}{\partial x}(a_u V_t V_x + a_u \psi_t V_x + a_u V_{tx})\cdot V_t \frac{\operatorname{sgn}_\eta |\operatorname{grad}V|}{|\operatorname{grad}V|} w_\lambda(x)\mathrm{d}x -$$

$$\int_0^n \frac{\partial}{\partial x}(f_V V_x + f_x) V_x \frac{\operatorname{sgn}_\eta |\operatorname{grad}V|}{|\operatorname{grad}V|} w_\lambda(x)\mathrm{d}x - \int_0^n \frac{\partial}{\partial x}(f_V V_t + f_t) V_t \frac{\operatorname{sgn}_\eta |\operatorname{grad}V|}{|\operatorname{grad}V|} w_\lambda(x)\mathrm{d}x -$$

$$\int_0^n \left[\left(\frac{\partial}{\partial x} g\right) V_x + \left(\frac{\partial}{\partial t} g\right) V_t\right] \frac{\operatorname{sgn}_\eta |\operatorname{grad}V|}{|\operatorname{grad}V|} w_\lambda(x)\mathrm{d}x = 0.$$

经分部积分，得 $\dfrac{d}{dt}\displaystyle\int_0^n I_\eta(\operatorname{grad}V) w_\lambda(x)\mathrm{d}x + \int_0^n a\left(\frac{\partial^2 I_\eta}{\partial \xi_1^2} V_{xx}^2 + 2\frac{\partial^2 I_\eta}{2\xi_1 2\xi_2} V_{xx} V_{xt} + \frac{\partial^2 I_\eta}{2\xi_2^2} V_{xt}^2\right) w_\lambda(x)\mathrm{d}x +$

$$\int_0^n a V_{xx} V_x \frac{\operatorname{sgn}_\eta |\operatorname{grad}V|}{|\operatorname{grad}V|} w_\lambda'(x)\mathrm{d}x + \int_0^n a V_{xt} V_t \frac{\operatorname{sgn}_\eta |\operatorname{grad}V|}{|\operatorname{grad}V|} w_\lambda'(x)\mathrm{d}x +$$

$$\int_0^n a_u \psi_x V_x \left(\frac{\partial^2 I_\eta}{\partial \xi_1 \partial \xi_2} V_{xt} + \frac{\partial^2 I_\eta}{\partial \xi_1^2} V_{xx}\right) w_\lambda(x)\mathrm{d}x + \int_0^n a_u \psi_t V_x \left(\frac{\partial^2 I_\eta}{\partial \xi_1 \partial \xi_2} V_{xx} + \frac{\partial^2 I_\eta}{\partial \xi_2^2} V_{tx}\right) w_\lambda(x)\mathrm{d}x +$$

$$\int_0^n a_u(\psi_x V_x + \psi_t V_t) V_x \frac{\operatorname{sgn}_\eta |\operatorname{grad}V|}{|\operatorname{grad}V|} w_\lambda'(x)\mathrm{d}x + \int_0^n a_u V_x I_\eta w_\lambda'(x)\mathrm{d}x + \int_0^n f_V I_\eta w_\lambda'(x)\mathrm{d}x -$$

$$\int_0^n \left(\frac{\partial}{\partial x} a_u\right) V_x (|\operatorname{grad}V| \operatorname{sgn}_\eta |\operatorname{grad}V| - I_\eta) w_\lambda(x)\mathrm{d}x - \int_0^n a_u V_{xx}(|\operatorname{grad}V| \operatorname{sgn}_\eta |\operatorname{grad}V| - I_\eta) w_\lambda(x)\mathrm{d}x$$

$$\int_0^n a_u V_{xx}(|\operatorname{grad}V| \operatorname{sgn}_\eta |\operatorname{grad}V| - I_\eta) w_\lambda(x)\mathrm{d}x - \int_0^n \left(\frac{\partial}{\partial x} f_V\right)(|\operatorname{grad}V| \operatorname{sgn}_\eta |\operatorname{grad}V| - I_\eta) w_\lambda(x)\mathrm{d}x -$$

$$a\frac{\partial I_\eta}{\partial x} w_\lambda \Big|_{x=0}^{x=n} - a_u V_x I_\eta w_\lambda \Big|_{x=0}^{x=n} - f_V I_\eta w_\lambda \Big|_{x=0}^{x=n} = \int_0^n \left(\frac{\partial}{\partial x} f_x + \frac{\partial}{\partial x} g\right) V_x \frac{\operatorname{sgn}_\eta |\operatorname{grad}V|}{|\operatorname{grad}V|} w_\lambda(x)\mathrm{d}x +$$

$$\int_0^n \left(\frac{\partial}{\partial x} f_t + \frac{\partial}{\partial t} g\right) V_t \frac{\operatorname{sgn}_\eta |\operatorname{grad}V|}{|\operatorname{grad}V|} w_\lambda(x)\mathrm{d}x. \qquad (2.8)$$

利用引理 2 和方程（2.5）可知，（2.8）式左端最后三项之和可以用 $\int_0^n |\operatorname{grad}V| w_\lambda(x)\mathrm{d}x$ 来估计．事实上，注意到 $u_t|_{x=0,n}=0$，我们有

$$a\frac{\partial I_\eta}{\partial x} w_\lambda \Big|_{x=0}^{x=n} + a_u V_x I_\eta w_\lambda(x) \Big|_{x=0}^{x=n} + f_V I_\eta w_\lambda \Big|_{x=0}^{x=n} =$$

$$a V_{xx} \frac{\operatorname{sgn}_\eta |\operatorname{grad}V|}{|\operatorname{grad}V|} V_x w_\lambda \Big|_{x=0}^{x=n} + a_u V_x I_\eta w_\lambda \Big|_{x=0}^{x=n} + f_V I_\eta w_\lambda \Big|_{x=0}^{x=n} =$$

$$-a_u V_x\left(\frac{\operatorname{sgn}_\eta|\operatorname{grad}V|}{|\operatorname{grad}V|}V_x^2-I_\eta\right)w_\lambda\Big|_{x=0}^{x=n}-a_u\psi_x V_x\frac{\operatorname{sgn}_\eta|\operatorname{grad}V|}{|\operatorname{grad}V|}V_x w_\lambda\Big|_{x=0}^{x=n}-$$

$$f_V\left(\frac{\operatorname{sgn}_\eta|\operatorname{grad}V|}{|\operatorname{grad}V|}V_x^2-I_\eta\right)w_\lambda\Big|_{x=0}^{x=n}-(f_x+g)\frac{\operatorname{sgn}_\eta|\operatorname{grad}V|}{|\operatorname{grad}V|}V_x w_\lambda\Big|_{x=0}^{x=n}$$

上式中右端第四项有界．利用引理 2，第二项可以用$\int_0^n|\operatorname{grad}V|w_\lambda(x)\mathrm{d}x$来估计．由于当$n\to0$时，$\left(\frac{\operatorname{sgn}_\eta|\operatorname{grad}V|}{|\operatorname{grad}V|}V_x^2-I_\eta\right)\Big|_{x=0}^{x=n}\to0$，从而，上式的第一项和第三项当$\eta\to0$时趋于零．

类似于文〔3〕中相应定理的证明可以证明：存在$\beta>0$，使

$$\int_0^n a\left(\frac{\partial^2 I_\eta}{\partial\xi_1^2}V_{xx}^2+2\frac{\partial^2 I_\eta}{\partial\xi_1\partial\xi_2}V_{xx}V_{xt}+\frac{\partial^2 I_\eta}{\partial\xi_2^2}V_{xt}^2\right)w_\lambda(x)\mathrm{d}x+\int_0^n a_u V_x\psi_x\left(\frac{\partial^2 I_\eta}{\partial\xi_1\partial\xi_2}V_{xt}+\frac{\partial^2 I_\eta}{\partial\xi_1^2}V_{xx}\right)w_\lambda(x)\mathrm{d}x+$$

$$\int_0^n a_u V_x\psi_t\left(\frac{\partial^2 I_\eta}{\partial\xi_1\partial\xi_2}V_{xx}+\frac{\partial^2 I_\eta}{\partial\xi_2^2}V_{tx}\right)w_\lambda(x)\mathrm{d}x\geq-\beta\int_0^n|\operatorname{grad}V|w_\lambda(x)\mathrm{d}x.$$ 又由

$$\int_0^n aV_{xx}V_x\frac{\operatorname{sgn}_\eta|\operatorname{grad}V|}{|\operatorname{grad}V|}w'_\lambda(x)\mathrm{d}x+\int_0^n aV_{xt}V_t\frac{\operatorname{sgn}_\eta|\operatorname{grad}V|}{|\operatorname{grad}V|}w'_\lambda(x)\mathrm{d}x+\int_0^n a_u V_x I_\eta w'_\lambda(x)\mathrm{d}x=$$

$$aI_\eta w'_\lambda(x)\Big|_{x=0}^{x=n}-\int_0^n a_u\psi_x I_\eta w'_\lambda(x)\mathrm{d}x-\int_0^n aI_\eta w''_\lambda(x)\mathrm{d}x$$ 及

$|w'_\lambda(x)|<\lambda w_\lambda(x)$，$|w''_\lambda(x)|\leq(\lambda+\lambda^2)w_\lambda(x)$．可知（2.8）左端第三项、第四项、第七项、第八项、第九项可以用$\int_0^n|\operatorname{grad}V|w_\lambda(x)\mathrm{d}x$来估计．再注意到当$\eta\to0$时、（2.8）式左端第十项、第十一项、第十二项趋于零，从而，在（2.8）式中令$\eta\to0$，有

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^n|\operatorname{grad}V|w_\lambda(x)\mathrm{d}x\leq C_3+C_4\int_0^n|\operatorname{grad}V|w_\lambda(x)\mathrm{d}x.$$

其中，$C_3$、$C_4$是与$n$无关的常数．再利用Gronwall引理，便得到$\int_0^n|\operatorname{grad}V|w_\lambda(x)\mathrm{d}x\leq M_2$.证毕．

**定理 1** 设存在$\eta>0$，使得当$|u|\leq M$（$M$是（2.4）中的常数）时$a(u)-\eta a_u^2(u)\geq0$，则问题（1.1）—（1.3）存在广义解．

**证明** 考虑问题（2.1）—（2.3），根据极值原理，在$\overline{Q}_T^n=\{(x,t);\ 0\leq x\leq n,\ 0\leq t\leq T\}$上 $0<u_{n+1}(x,t)\leq u_n(x,t)$．

因此，我们可以在$Q_T$上定义一个实的非负有界函数$u(x,t)$：$u(x,t)=\lim\limits_{n\to\infty}u_n(x,t)$．由文〔1〕的证明可知$u(x,t)$连续，利用命题的结论容易验证$u\in BV(Q_T)$．事实上

$$\int_0^\infty|u(x+h_1,t+h_2)-u(x,t)|w_\lambda(x)\mathrm{d}x=\lim_{n\to\infty}\int_0^\infty|u_n(x+h_1,t+h_2)-u_n(x,t)|w_\lambda(x)\mathrm{d}x\leq$$

$$\lim_{n\to\infty}\left[\int_0^\infty|u_n(x+h_1,t+h_2)-u_n(x+h_1,t)|w_\lambda(x)\mathrm{d}x+\int_0^\infty|u_n(x+h_1,t)-u_n(x,t)|w_\lambda(x)\mathrm{d}x\right]\leq$$

$$\lim_{n\to\infty}\left[\int_0^\infty\int_0^1\left|\frac{\partial u_n(x+h_1,t+\tau h_2)}{\partial t}\right|w_\lambda(x)|h_2|\mathrm{d}\tau\mathrm{d}x+\int_0^\infty\int_0^1\left|\frac{\partial u_n(x+\tau h_1,t)}{\partial x}\right|w_\lambda(x)|h_1|\mathrm{d}\tau\mathrm{d}x\right]\leq$$

$N|h|$，其中，$h=(h_1,h_2)\in Q_T$，$|h|=(h_1^2+h_2^2)^{1/2}$．

下面证明$u(x,t)$满足（1.4）．对§1广义解的定义3中的任意给定的$\varphi(x,t)$，取$m$充分大，使$\varphi(x,t)=0$当$x\geq m$，$0\leq t\leq T$．则对$n\geq m$有

$$\iint_{Q_T^n}[u_n\varphi_t+A(u_n)\varphi_{xx}-B(u_n)\varphi_x]\mathrm{d}x\mathrm{d}t=-\int_0^m u_{0n}(x)\varphi(x,0)\mathrm{d}x+\int_0^T A(M)\varphi_x(m,t)\mathrm{d}t-$$

$\int_0^T A(\psi_n(t))\varphi_x(0, t)\mathrm{d}t$，式中$Q_T^m=\{(x,t): 0<x<m, 0<t<T\}$．因么此，在上式中先令$n\to\infty$，再令$m\to\infty$，便得到（1.1）式．

## §3 广义解的唯一性

**引理 3** 设$u_1$、$u_2$是问题（1.1）—（1.3）的广义解，则有积分不等式

$$\iint_{Q_T}\left\{|u_1-u_2|\varphi_t-\mathrm{sgn}(u_1-u_2)\left[a(u_1)\frac{\partial u_1}{\partial x}-a(u_2)\frac{\partial u_2}{\partial x}+(B(u_1)-B(u_2)\right]\varphi_x\right\}\mathrm{d}x\mathrm{d}t\geq 0. \quad (2.9)$$

式中，$\varphi\in C^1(\overline{Q}_T)$，$\varphi\geq 0$；$\varphi(x, 0)=\varphi(x, T)=0$；当$x$充分大时$\varphi(x, t)=0$．

引理 3 的证明与文〔3〕中的定理（2.1）相同．

**定理 2** 由定理 1 给出的问题（1.1）—（1.3）的广义解是唯一的．

**证明** 注意到$\mathrm{sgn}(u_1-u_2)(A(u_1)-A(u_2))\in BV$，且

$$\frac{\partial}{\partial x}\left[\mathrm{sgn}(u_1-u_2)\cdot(A(u_1)-A(u_2))\right]=\mathrm{sgn}(u_1-u_2)\cdot\frac{\partial}{\partial x}(A(u_1)-A(u_2))$$

对（2.9）分部积分，得

$$\iint_{Q_T}\left\{|u_1-u_2|\varphi_t-\mathrm{sgn}(u_1-u_2)\cdot(B(u_1)-B(u_2))\varphi_x+\mathrm{sgn}(u_1-u_2)(A(u_1)-A(u_2))\varphi_{xx}\right\}\mathrm{d}x\mathrm{d}t\geq 0.$$

取$\varphi(x, t)=w_\lambda(x)(\rho_h(t-s)-\rho_h(t-\tau))$，$0<s<\tau<T$．其中$\rho_h(\sigma)=\int_{-\infty}^{\sigma}\delta_h(\tau)\mathrm{d}\tau$；$\delta(\tau)\geq 0$；$\delta(\tau)=0$（$|\tau|\geq 1$）；$\int_{-\infty}^{\infty}\delta(\tau)\mathrm{d}\tau=1$；$\delta_h(\tau)=\frac{1}{h}\delta(\frac{\tau}{h})$．在上面不等式中令$h\to 0$，并注意到$|w'_\lambda(x)|\leq\lambda w_\lambda(x)$，$|w''_\lambda(x)|\leq(\lambda+\lambda^2)w_\lambda(x)$，有

$$\int_0^\infty|u_1(x, \tau)-u_2(x, \tau)|w_\lambda(x)\mathrm{d}x-\int_0^\infty|u_1(x, s)-u_2(x, s)|w_\lambda(x)\mathrm{d}x\leq N\int_s^\tau\int_0^\infty|u_1-u_2|w_\lambda\mathrm{d}x.$$

再令$s\to 0$，有

$$\int_0^\infty|u_1(x, \tau)-u_2(x, \tau)|w_\lambda(x)\mathrm{d}x\leq N\int_0^\tau\int_0^\infty|u_1(x, t)-u_2(x, t)|w_\lambda(x)\mathrm{d}x$$

由此及Gronwall引理，便得到了所要证明的结果．

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