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The Logical Derivatives and Integrals (Ⅱ) *

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The logical (or p-adic) derivative and integral of a complex function on $R^+=[0,\infty)$ are defined in [2], [4]. Some relations between p-adic derivative $D^{\langle 1\rangle}f$ and integral $I^{\langle 1\rangle}f$ for $f\in L^q_{[0,1)}$, $1\le q<\infty$, and for $f\in L^q_{R^+}$, $1\le q\le 2$, are discussed, see also [1], [5], e.g., for some functions f, one has the formulas:

$$D^{\langle 1\rangle}(I^{\langle 1\rangle}f)=f, \qquad I^{\langle 1\rangle}(D^{\langle 1\rangle}f)=f.$$

In this note we are continuing the discussion of [4]. In particular, we define the Walsh-Fourier Transform (WFT) of $f\in L^q_{R^+}$, $2<q<\infty$ and extend some results on $D^{\langle 1\rangle}f$ and $I^{\langle 1\rangle}f$ by using the test function class and distribution theory.

I Preliminaries

For any integer $N\in\mathbf{Z}$, let

$$I_{k,N}=\{x\in R^+:\ kp^{-N}\le x<(k+1)p^{-N}\},\quad k\in\mathbf{P}=\{0,1,2,\cdots\}.$$

Denote by $\Phi_N(x)$ the characteristic functions of the interval $I_{0,N}=[0,p^{-N})$, τ_h the translation operator $(\tau_h f)(x)=f(x\ominus h)$, $x,h\in R^+$, and U the class

$$U=\{\varphi:\ \varphi(x)=\sum_{j=0}^{n}c_j\tau_{h_j}\Phi_N(x),\quad c_j\in\mathbf{C},h_j\in R^+,n\in\mathbf{N}\}.$$

It is clear that each $\varphi\in U$ has compact support.

We call $\{\varphi_n\}$ in U a null sequence, if (i) there is a fixed pair of integers N,s such that each φ_n is a constant on any interval $I_{j,N}$ and is supported on the compact set $\bar{I}_{0,s}$, and (ii) $\lim_{n\to\infty}\varphi_n(x)=0$ uniformly on R^+. With this topology, U becomes a topological linear space over $\mathbf{C}$, and it is obviously Hausdorff and complete. We call U the test function class. As usual, with the weak* topology the collection U^* of all continuous linear functionals on U is said to be the space of distributions. The action of $f\in U^*$ on $\varphi\in U$ is denoted by (f,φ). Let $\varphi^\wedge$ be the WFT of $\varphi\in U$, it is easy to see that the WFT is a homeomorphism on U.

The WFT of $f\in U$ is defined by the formula:

$$(f^\wedge,\varphi)=(f,\varphi^\wedge),\ \varphi\in U.$$

Convolutions, products and invese WFT can be defined in the usual way. See [3] for details.

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Let f be a function in $L^q_{R^+}$, $1 \leq q < \infty$. We define its WFT $f^\wedge$, when it exists, to be a distribution, such that the equalities

$$(f^\wedge, \varphi) = (f, \varphi^\wedge), \quad \forall \varphi \in U$$

are fulfilled. In this case, we always assume that $f^\wedge$ is a linear functional over $L^{q'}_{R^+}$, where q' is the conjugate index of q so that the domain of $f^\wedge$ can be extended from U to the whole space $L^{q'}_{R^+}$.

If $f, g \in L^q_{R^+}$, $1 \leq q < \infty$, the product $f^\wedge g$ is defind as a distribution by

$$(f^\wedge g, \varphi) = (f^\wedge, g\varphi), \quad \forall \varphi \in U.$$

If $f \in L^q_{R^+}$, $1 \leq q < \infty$, $\psi \in U$, the convolution $f * \psi$ is a functional h over $L^q_{R^+}$, satisfying

$$(h, \varphi) = (f, \tilde{\psi} \circledast \varphi), \quad \forall \varphi \in U,$$

where $\tilde{\psi}(x) = \psi(-x)$, and $(\psi \circledast \varphi)(x) = \int_{R^+} \psi(y)\varphi(x \ominus y)dy$, and as we see, $\tilde{\psi} \circledast \varphi$ is again in U. This definition can be generalized to the case of $f \in L^q_{R^+}$, $g \in L^{q'}_{R^+}$. Then in the distribution sense we have $(f \circledast g)^\wedge = f^\wedge g^\wedge$.

2 The p-adic derivatives and integals of $f \in L^q_{R^+}$, $2 < q < \infty$

We use the following notations (compare with [4])

$$D_a(t) = \int_0^a \omega(x, t)dx, \quad a, t \in R^+,$$
$$S(f, a; x) = \int_0^{\to\infty} f(x \ominus u) D_a(u) du \equiv (D_a \circledast f)(x)$$

and

$$(D_m^{\langle 1 \rangle} f)(x) = \sum_{k=-m}^{m} p^k \sum_{j=0}^{p-1} A_j f(x \oplus j p^{-k-1}).$$

$D^{\langle 1 \rangle} f$ is the p-adic derivative of f, it is the strong limit of $D_m^{\langle 1 \rangle} f$ in $L^q_{R^+}$.

$V_{1,n}$ is a basic function defined by its WFT

$$V^\wedge_{1,n}(t) = \begin{cases} \frac{1}{t}, & t \in [p^{-n}, \infty), \\ 0, & t \in [0, p^{-n}), \end{cases} \qquad n \in Z,$$

and we have $V_{1,n} \in L^1_{R^+} \cap L^2_{R^+}$ for each fixed n [4].

$I^{\langle 1 \rangle} f$ is the p-adic integral of f, it is the strong limit of $(V_{1,n} \circledast f)(x)$ in $L^q_{R^+}$.

For the WFT of p-adic derivative, we have

Theorem 1 If f and $D^{\langle r \rangle} f \in L^q_{R^+}$, then for $r \in N$, $[D^{\langle r \rangle} f]^\wedge = t^r f^\wedge$ in the distribution sense.

Proof We only deal with the case $r = 1$; $r > 1$ can be completed by induction.

For $r = 1$, by definition, the table [5] of WFT and the Lebesgue dominated convergent theorem, it follows for all $\varphi \in U$

$$([D^{\langle 1 \rangle} f]^\wedge \varphi) = (D^{\langle 1 \rangle} f, \varphi^\wedge) = \lim_{m \to \infty} \sum_{j=0}^{p-1} \left(\sum_{k=-m}^{m} p^k A_j f, \tau_{jp^{-k-1}} \varphi^\wedge \right)$$
$$= \lim_{m \to \infty} \left(\left[\sum_{k=-m}^{m} p^k \sum_{j=0}^{p-1} A_j w(jp^{-k-1}, \circ) \varphi(\circ) \right]^\wedge, \right.$$
$$= \left(\left[\lim_{m \to \infty} \sum_{k=-m}^{m} p^k \sum_{j=0}^{p-1} A_j w(jp^{-k-1}, \circ) \varphi(\circ) \right]^\wedge, f \right)$$

$$=([\circ\varphi(\circ)]^{\wedge}, f)=(f^{\wedge}, \varphi(\circ))=(\circ f^{\wedge}(\circ), \varphi)$$

Therefore $[D^{\langle 1\rangle}f]^{\wedge}=v f^{\wedge}$ in the distribution sense.

To establish the main theorem, we need a series of lemmas.

Lemma 1 [4] If $f\in L^q_{R^+}$, then $S(f;p^n;x)\in L^q_{R^+}$ and

$$\|S(f;p^n;\circ)\|_q\leq\|f\|_q,\quad \lim_{n\to\infty}\|S(f;p^n;\circ)-f(\circ)\|_q=0.$$

Lemma 2 If $f\in L^q_{R^+}$ and $f^{\wedge}=0$, then $f(x)=0$ $a.e.$

Proof For all $\varphi\in U$, we have $(f,\varphi^{\wedge})=(f^{\wedge},\varphi)=0$ and since FWT is a homeomorphism on U, we conclude $f(x)=0$ $a.e.$

Lemma 3 If $f\in L^q_{R^+}$, and $\lim_{n\to\infty}\int_0^{p^n}f(u)du=0$, then $\lim_{n\to\infty}\|S(f;p^{-n},\circ)\|_q=0$.

Proof Let $\varepsilon>0$ be given, choose $n_\varepsilon>0$, such that both of the inequalities

$$\left|\int_0^{p^n}f(u)du\right|<(\varepsilon/2)^{1/q},\quad \int_{p^n}^{\infty}|f(u)|^q du<(\varepsilon/2)^{1/q}$$

hold for $n>n_\varepsilon$. And since

$$S(f;p^{-n};x)=\int_0^{\infty}f(x\ominus u)D_{p^{-n}}(u)du=p^{-n}\int_0^{p^n}f(x\ominus u)du,$$

we have for $n>n_\varepsilon$

$$\|S(f;p^{-n};\circ)\|_q^q=\int_0^{p^n}\left|p^{-n}\int_{\infty}^{p^n}f(x\ominus u)du\right|^q dx=\left\{\int_0^{p^n}+\int_0^{\infty}\right\}\left|p^n\int_0 f(x\ominus u)du\right|^q dx=I_1+I_2,$$

say. For I_1, it follows

$$I_1^{1/q}=\left\{\int_0^{p^n}\left|p^{-n}\int^{p^n}f(x\ominus u)du\right|^q dx\right\}^{1/q}<\left\{\frac{\varepsilon}{2}p^{-n}\int_0^{p^n}dx\right\}^{1/q}=\left(\frac{\varepsilon}{2}\right)^{1/q},$$

so that $I_1<\varepsilon/2$ as $n>n_\varepsilon$. For I_2, we have

$$I_2^{1/q}=\left\{\int_{p^n}^{\infty}p^{-n}\int^{p^n}f(x\ominus u)du\,|^q dx\right\}^{1/q}\leq p^{-n}\int_0^{p^n}\left\{|f(x)|^q dx\right\}^{1/q}du,$$

therefore $I_2<\varepsilon/2$ for $n>n_\varepsilon$, the proof is complete.

Lemma 4 If $f\in L^q_{R^+}$, and $D^{\langle 1\rangle}f=0$, then $f(x)=0$ $a.e.$

Proof We have for all $\varphi\in U$

$$([D^{\langle 1\rangle}f]^{\wedge},\varphi)=(D^{\langle 1\rangle}f,\varphi^{\wedge})=0,$$

hence $[D^{\langle 1\rangle}f]^{\wedge}=0$ in the distribution sense. By Theorem 1, $vf^{\wedge}=[D^{\langle 1\rangle}f]^{\wedge}=0$ $a.e.$ It means for any $\varphi\in U$, $(f^{\wedge},v\varphi)=(vf^{\wedge},\varphi)=0$. On the other hand, it is plain that the class $vU=\{v\varphi;\varphi\in U\}$ is dense in $L^q_{R^+}$, so $f^{\wedge}=0$ as a distribution. Hence $f(x)=0$ $a.e.$ again by Theorem 1. That is all for the proof.

Lemma 5 Let $m,n\in N$, and $m>n$. Then

$$(V_{1,n}\otimes f)(x)-(V_{1,m}\otimes f)(x)=S(g;p^{-n};x)-S(g;p^{-m};x)\qquad a.e.$$

Where $f,g\in L^q_{R^+}$ and $g^{\wedge}=\begin{cases}v^{-1}f, & v\in(0,\infty),\\ 0, & v=0.\end{cases}$

Proof It is well known that $V_{1,n}\in L^1_{R^+}$, whence

$$[V_{1,n}\otimes f]^{\wedge}=V_{1,n}^{\wedge}f^{\wedge}=\begin{cases}v^{-1}f^{\wedge}, & v\in[p^{-n},\infty),\\ 0, & v\in[0,p^{-n}).\end{cases}$$

Therefore for $m>n$, it follows

$$[V_{1,n}\otimes f]^\wedge-[V_{1,m}\otimes f]^\wedge=\begin{cases}0, & v\in[p^{-n},\infty),\\ v^{-1}f, & v\in[p^{-m},p^{-n})\\ 0, & v\in[0,p^{-n}).\end{cases}$$

On the other hand,

$$[S(g;p^{-n};\circ)]^\wedge(v)=g^\wedge D^\wedge_{p^{-n}}(v)=\begin{cases}0, & v\in[p^{-n},\infty),\\ g^\wedge, & v\in[0,p^{-n}),\end{cases}$$

so

$$[S(g;p^{-n};\circ)]^\wedge(v)-[S(g;p^{-m};\circ)]^\wedge(v)=\begin{cases}0, & v\in[p^{-n},\infty),\\ g^\wedge, & v\in[p^{-m},p^{-n}),\\ 0, & v\in[0,p^{-m}).\end{cases}$$

By Lemma 2 we get the formula

$$(V_{1,n}\otimes f)(x)-(V_{1,m}\otimes f)(x)=S(g;p^{-n};x)-S(g;p^{-m};x)\ a.e.$$

The Lemma is proved.

Now let $\widetilde{U}$ be the subclass of U:

$$\widetilde{U}=\{\varphi\in U:\int_{R^+}\varphi(t)dt=0\}.$$

For our purpose we would like to introduce a condition as follows.

Condition(*) If $g\in L^q_{R^+}$, and $g^\wedge$ can be determined uniquely by $(g^\wedge,\psi^\wedge)=(g,\tilde{\psi})$ for every $\psi\in\widetilde{U}$, and where $\tilde{\psi}(x)=\psi(-x)$, then we say th t g satisfies Condition (*).

Lemma 6 For all $\varphi\in U,\varphi^\wedge$ vanishes in some neighbourhood of 0.

Proof Let $\varphi\in U$, and

$$\varphi(x)=\sum_{j=0}^{n}c_j\tau_{h_j}\Phi_N(x),\quad c_j\in\mathbf{C},\ h_j\in\mathbf{R}^+,\ n\in\mathbf{N},$$

it follows by [5]

$$\varphi^\wedge(t)=\begin{cases}\sum_{j=0}^{n}p^{-N}c_j\bar{\omega}(h_j,t), & 0\le t<p^N,\\ 0, & t\ge p^N.\end{cases}$$

Since $\int_{R^+}\varphi(t)dt=0$ implies $\sum_{j=0}^{n}c_j=0$, we conclude that $\varphi^\wedge(t)$ is equal to zero in a neighbourhood of 0.

Lemma 7 Let $f\in L^q_{R^+}$ and $I^{\langle 1\rangle}f$ exist in $L^q_{R^+}$-sense. Assume that $g=I^{\langle 1\rangle}f$ satisfies Condition (*), then the formula $g^\wedge=v^{-1}f^\wedge$ holds in the distribution sense.

Proof Since $g=I^{\langle 1\rangle}f$, by definition, $\|g-V_{1,n}\otimes f\|_q\to 0,\ n\to\infty$. Note that strong convergence implies weak convergence, so for all $\varphi\in U$, we have

$$(g^\wedge-[V_{1,n}\otimes f]^\wedge,\varphi)=(g-V_{1,n}\otimes f,\hat{\varphi})\to 0,\ n\to\infty,$$

hence

$$\lim_{n\to\infty}(g^\wedge-[V_{1,n}\otimes f]^\wedge,\varphi)=\lim_{n\to\infty}(g^\wedge-V^\wedge_{1,n}f^\wedge,\varphi)=0,\quad\forall\varphi\in U$$

and

$$\lim_{n\to\infty}(V^\wedge_{1,n}f^\wedge,\varphi)=(g^\wedge,\varphi),\quad\forall\varphi\in U.$$

In virtue of $V^\wedge_{1,n}\varphi\in L^q_{R^+}$ and g satisfies Condition (*), we can apply

$$\lim_{n\to\infty}(V'_{1,n}f',\psi')=(g',\psi'),\quad \psi\in\widetilde{U}$$

instead of

$$\lim_{n\to\infty}(V'_{1,n}f',\varphi)=(g',\varphi),\quad \varphi\in U.$$

But by Lemma 6, for all sufficient large n, it follows $V'_{1,n}\psi'=\frac{1}{v}\psi'$, this gives

$$\lim_{n\to\infty}(V^{\wedge}_{1,n}f^{\wedge},\psi^{\wedge})=\lim_{n\to\infty}(f^{\wedge},V^{\wedge}_{1,n}\psi^{\wedge})=(f',\frac{1}{v}\psi')=(\frac{1}{v}f',\psi'),\quad \psi\in\widetilde{U}.$$

Therefore $(\frac{1}{v}f^{\wedge},\psi^{\wedge})=(g^{\wedge},\psi^{\wedge})$ for all $\psi\in\widetilde{U}$. Thus we have $v^{-1}f=g'$.

Lemma 8 Let $f\in L^q_{R^+}$. Assume that $D^{\langle 1\rangle}f$ exists in $L^q_{R^+}$-sense. Then we have

$$\lim_{m\to\infty}\int_0^{p^m}D^{\langle 1\rangle}f(t)\,dt=0.$$

Proof For any $m\in\mathbf{N}$, we have by dominated convergence theorem

$$\lim_{N\to\infty}\int_0^{p^m}\sum_{k=-N}^{N}p^k\sum_{j=0}^{p-1}A_jf(t\oplus jp^{-k-1})\,dt=\int_0^{p^m}D^{\langle 1\rangle}f(t)\,dt.$$

On the other hand, if $N>m$,

$$\int_0^{p^m}\sum_{k=-N}^{N}p^k\sum_{j=0}^{p-1}A_jf(t\oplus jp^{-k-1})\,dt=\sum_{k=-N}^{N}\left\{p^k\sum_{j=0}^{p-1}A_j\int_0^{p^m}f(t\oplus jp^{-k-1})\,dt\right\}$$

$$=\sum_{k=-N}^{-m-1}+\sum_{k=-m}^{N}=I_{m,N}+J_{m,N}$$

say. We assert $J_{m,N}=0$. In fact, for $k=-m,-m+1,\cdots,N$, $j=0,1,\cdots,p-1$, each transform $t\to t\oplus jp^{-k-1}$ is a one-one mapping on $[0,p^m)$, saving for a denumerable set, [6], hence the integrals $\int_0^{p^m}f(t\oplus jp^{-k-1})\,dt$ take the same value, and in virtue of $A_0+A_1+\cdots+A_{p-1}=0$ [6], the conclusion follows. To estimate $I_{m,N}$, we use Hölder inequality

$$|I_{m,N}|\le\sum_{k=-N}^{-m-1}p^k\sum_{j=0}^{p-1}|A_j|\,\|f\|_q\,p^{m/q'},$$

where q' is the conjugate index of q $(q'>1)$. Thus

$$|I_{m,N}|\le\|f\|_q\sum_{j=0}^{p-1}|A_j|\,(p-1)^{-1}p^{-m/q}.$$

Note that the right hand side is independent of N, and is $O(p^{-m/q})$, so $\lim_{m\to\infty}\lim_{N\to\infty}I_{m,N}=0$, thus $\lim_{m\to\infty}\int_0^{p^m}D^{\langle 1\rangle}f(t)\,dt=0$. The Lemma is proved.

Lemma 9 Let $f,g\in L^q_{R^+}$. If $\lim_{m\to\infty}\int_0^{p^m}g(u)\,du=0$, and $g^{\wedge}=v^{-1}f^{\wedge}$, then $g=I^{\langle 1\rangle}f$.

Proof By the formula in Lemma 5

$$(V_{1,n}\circledast f)(x)-(V_{1,m}\circledast f)(x)=S(g;p^{-n};x)-S(g;p^{-m};x)\quad a.e.$$

From Lemma 3 it follows

$$\lim_{m,n\to\infty}\|(V_{1,n}\circledast f)(\circ)-(V_{1,m}\circledast f)(\circ)\|_q=0.$$

By the completeness of $L^q_{R^+}$, there exists $h\in L^q_{R^+}$, such that

$$\lim_{n\to\infty}\|h(\circ)-(V_{1,n}\circledast f)(\circ)\|_q=0,$$

and we have $h=I^{\langle 1\rangle}f$. Setting $h_n=V_{1,n}\circledast f$, it follows for all $\varphi=\psi^{\wedge}$, $\psi\in\widetilde{U}$

$$(h_n^{\wedge},\varphi)=([V_{1,n}\circledast f]^{\wedge},\varphi)=(V^{\wedge}_{1,n}f^{\wedge},\varphi)=(f^{\wedge},V^{\wedge}_{1,n}\varphi).$$

Thus for all sufficient large n, one has

$$(f^{\wedge},V^{\wedge}_{1,n}\varphi)=(f^{\wedge},\frac{1}{v}\varphi)=(\frac{1}{v}f^{\wedge},\varphi)=(g^{\wedge},\varphi).$$

But $\lim\limits_{n\to\infty}(h_n^\wedge, \varphi) = (h^\wedge, \varphi)$, we conclude that $g^\wedge = h^\wedge$ and consequently $g = I^{\langle 1\rangle} f$.

Now we turn to study some relations between p-adic derivative and p-adic integral.

Theorem 2 If $f, D^{\langle 1\rangle} f \in L^q_{R^+}$, $r \in \mathbf{N}$, and f satisfies condition $(*)$, then $f = I^{\langle r\rangle}(D^{\langle r\rangle} f)$.

Proof By Theorem 1 the equality $r=1$, $[D^{\langle 1\rangle} f]^\wedge = v f^\wedge$ is valid in the distribution sence. Furthermore, for all $\varphi = \psi^\wedge$ with $\psi \in \widetilde{U}$, we have

$$(f^\wedge, \varphi) = (v f^\wedge, v^{-1}\varphi) = ([D^{\langle 1\rangle} f]^\wedge, v^{-1}\varphi) = (v^{-1}[D^{\langle 1\rangle} f]^\wedge, \varphi),$$

hence the assumption on f implies $f^\wedge = v^{-1}[D^{\langle 1\rangle} f]^\wedge$. By Lemma 8, we get $\lim\limits_{n\to\infty}\int_0^{p^n} D^{\langle 1\rangle} f(t)\mathrm{d}t = 0$, and consequently the formula $f = I^{\langle 1\rangle}(D^{\langle 1\rangle} f)$ by applying Lemma 9.

For $r > 1$ it can be done by induction.

Theorem 3 Let $f, g \in L^q_{R^+}$, if $g = I^{\langle 1\rangle} f$, $r \in \mathbf{N}$, such that $\lim\limits_{n\to\infty}\int_0^{p^n} g(u)\mathrm{d}u = 0$, and that g satisfies the Condition $(*)$. Then $f = D^{\langle 1\rangle}(I^{\langle r\rangle} f)$.

Proof We will show this formula by three steps for $r = 1$.

First step. We prove the inequality

$$g(x) = S(g; p^m; x) + (V_{1,-m} \circledast f)(x) \quad a.e. \tag{1}$$

for $m \in \mathbf{N}$. Since $g = I^{\langle 1\rangle} f \in L^q_{R^+}$, by Lemma 7, for $v \neq 0$ $g^\wedge = \frac{1}{v} f^\wedge$. In view of the definition of p-adic integral $\lim\limits_{n\to\infty}\| g(\quad) - (V_{1,n} \circledast f)(\circ)\|_q = 0$, there exists a subsequence $n_k \to \infty$, such that

$$\lim_{n\to\infty}(V_{1,n_k} \circledast f)(x) = g(x) \quad a.e.$$

Using the method of WFT for $n_k > m$, one has

$$(V_{1,n_k} \circledast f)(x) = S(V_{1,n_k} \circledast f;\ p^m;\ x) + (V_{1,-m} \circledast f)(x) \quad a.e. \tag{2}$$

Note that $D_{p^m} \in L^{q'}_{R^+}$ for every $2 < q < \infty$, so

$$\lim_{k\to\infty} S(V_{1,n_k} \circledast f;\ p^m;\ x) = \int_0^\infty g(x \ominus u) D_{p^m}(u)\mathrm{d}u = S(g;\ p^m;\ x) \qquad a.e.$$

Taking limit in (2) we have $g(x) = S(g; p^m; x) + (V_{1,-m} \circledast f)(x)\, a.e.$ This is (1).

By the way, from (1) it follows

$$(D_m^{\langle 1\rangle} g)(x) = D_m^{\langle 1\rangle} S(g;\ p^m;\ x) + D_m^{\langle 1\rangle}(V_{1,-m} \circledast f)(x)\ a.e. \tag{3}$$

Therefore

$$\| D_m^{\langle 1\rangle} g - f\|_q \le \| D_m^{\langle 1\rangle} S(g;\ p^m;\ \circ) - f(\circ)\|_q + \| D_m^{\langle 1\rangle}(V_{1,-m} \circledast f)(\circ)\|_q \tag{4}$$

Second step. We want to prove

$$\lim_{m\to\infty}\| D_m^{\langle 1\rangle} S(g;\ p^m;\ \circ) - f(\circ)\|_q = 0. \tag{5}$$

It follows by the method of WFT

$$S(f;\ p^m;\ x) = \sum_{k=-\infty}^{m-1} p^k \sum_{j=0}^{p-1} A_j,\ S(g;\ p^m;\ x \oplus j p^{-k-1})\ a.e.$$

Let

$$S(f,\ p^m;\ x) = \Big\{\sum_{k=-\infty}^{-m} + \sum_{k=-m+1}^{m-1}\Big\} p^k \sum_{j=0}^{p-1} A_j\, S(g;\ p^m;\ x \oplus j p^{-k-1}) \equiv J_1 + J_2$$

say. For J_1 we have

$$\| J_1\|_q \le \sum_{k=-\infty}^{-m} p^k \sum_{j=0}^{p-1} |A_j|\, \| s(g;\ p^m;\ \circ \oplus j p^{-k-1})\|_q \le \sum_{j=0}^{p-1} |A_j|\, \| g\|_q\, p^{-m+1},$$

hence $\lim\limits_{m\to\infty} \|J_1\|_q = 0$.

On the other hand, for $J_2 = D_m^{\langle 1\rangle} S(g;\ p^m;\ x)$, we get

$$\|S(f;\ p^m;\ \circ\ ) - J_2\|_q = \|S(f;\ p^m;\ \circ\ ) - D_m^{\langle 1\rangle} S(g;\ p^m;\ \circ\ )\|_q = \|J_1\|_q \to 0,\ m\to\infty, \quad (6)$$

and

$$\|D_m^{\langle 1\rangle}\ (g;\ p^m;\ \circ\ ) - f(\ \circ\ )\|_q \le \|D_m^{\langle 1\rangle} S(g;\ p^m;\ \circ\ ) - S(f;\ p^m;\ \circ\ )\|_q + \|S(f,\ p^m;\ \circ\ ) - f(\ \circ\ )\|_q$$

so by (6) and Lemma 1, it follows

$$\lim_{m\to\infty} \|D_m^{\langle 1\rangle}\ (g;\ p^m;\ \circ\ ) - f(\ \circ\ )\|_q = 0.$$

Third step. We prove $\lim\limits_{m\to\infty} \|D_m^{\langle 1\rangle}(V_{1,-m}\circledast f)(\ \circ\ )\|_q = 0$. It is obvious $V_{1,-m}\in L^1_{\mathbf{R}^+}$, $f\in L^q_{\mathbf{R}^+}$, so that $V_{1,-m}\circledast f\in L^q_{\mathbf{R}^+}$, $D_m^{\langle 1\rangle}(V_{1,-m}\circledast f)\in L^q_{\mathbf{R}^+}$. Then by the convolution theorem and the formula of derivatives $[D_m^{\langle 1\rangle}(V_{1,-m}\circledast f)] = [f\circledast D_m^{\langle 1\rangle} V_{1,-m}]$, thus by the uniqueness theorem

$$D_m^{\langle 1\rangle}(V_{1,-m}\circledast f)(x) = (f\circledast D_m^{\langle 1\rangle} V_{1,-m})(x) \quad a.e. \qquad (7)$$

Setting $G_m(x) = (D_m^{\langle 1\rangle} V_{1,-m})(x)$ and by Lemma 2.3 in [4], $\|V_{1,-m}\|_1 \le K p^{-m}$, K is a constant, one can conclude

$$\|G_m\|_1 = \Big\|\sum_{k=-(m-1)}^{m-1} p^k \sum_{j=0}^{p-1} A_j V_{1,-m}(\circ\oplus jp^{-k-1})\Big\|_1 \le \Big(\sum_{j=0}^{p-1}|A_j|\Big)\sum_{k=-(m-1)}^{m-1} p^k \|V_{1,-m}\|_1$$

$$\le \Big\{K p^{-m}\sum_{k=-(m-1)}^{m-1} p^k\Big\}\cdot\sum_{j=0}^{p-1}|A_j| \le M = \text{const}.$$

Therefore

$$\|f\circledast G_m\|_q \le \|G_m\circledast(f - S(f;\ p^k;\ \circ\ )\|_q + \|G_m\circledast S(f;\ p^k;\ \circ\ )\|_q$$

$$\le \|G_m\|_1\ \|S(f,\ p^k;\ \circ\ ) - f(\ \circ\ )\|_q + \|G_m\circledast S(f;\ p^k;\ \circ\ )\|_q. \qquad (8)$$

The first term of the right hand side in (8) tends to 0 by Lemma 1, and the second term is 0 when $m > k$, thus we have $\lim\limits_{m\to\infty}\|D_m^{\langle 1\rangle}(V_{1,-m}\circledast f)(\ \circ\ )\|_q = 0$, which is (7).

Now by (4), (5) and (7), we get $D^{\langle 1\rangle}(g = f\ a.e.$ By hypothesis $g = I^{\langle 1\rangle} f$, it follows, $D^{\langle 1\rangle}(I^{\langle 1\rangle} f) = f\ \ a.e.$ This proves the theorem for $r = 1$. The general case $r > 1$ is then done by induction.

The following theorem shows $D^{\langle r\rangle}$ is a closed operator.

Theorem 4 Let $r\in\mathbf{P}$. Denote by W_r the class

$$W_r = \{f\in L^q_{\mathbf{R}^+}:\ \exists D^{\langle 1\rangle} f\in L^q_{\mathbf{R}^+}\ \lim_{n\to\infty}\int_0^{p^n} f(t)\mathrm{d}t = 0 \text{ and } f \text{ is with Condition } (*)\}$$

Then $D^{\langle r\rangle}$ is a closed linear operator over W_r.

Proof Let $r = 1$, and we take f_n, f, g satisfying the following conditions:

i) $f_n\in W_r$

ii) $f, g\in L^q_{\mathbf{R}^+}$ and f, g are with Condition (*), and $\lim\limits_{m\to\infty}\int_0^{p^m} f(t)\mathrm{d}t = 0$,

iii) $\lim\limits_{n\to\infty}\|f_n - f\|_q = 0$, $\lim\limits_{n\to\infty}\|D^{\langle 1\rangle} f_n - g\|_q = 0$.

Then we have to prove $f\in W_r$ and $g = D^{\langle 1\rangle} f$.

In fact, since $\lim\limits_{n\to\infty}\|D^{\langle 1\rangle} f_n - g\|_q = 0$, we have for all $\varphi = \psi^\wedge$ with $\psi\in\widetilde{U}$

$$\lim_{n\to\infty}([D^{\langle 1\rangle} f_n]^\wedge - g^\wedge,\ \varphi) = 0.$$

But $[D_n^{\langle 1\rangle} f_n]^\wedge = v f_n^\wedge$, one has

$$0=\lim_{n\to\infty}([D^{\langle 1\rangle}f_n]^\wedge-g^\wedge,\varphi)=\lim_{n\to\infty}(vf_n^\wedge-g^\wedge,\varphi),\ \varphi=\psi^\wedge,\forall\psi\in\widetilde{U},$$

therefore by ii)

$$(vf^\wedge,\varphi)=(g^\wedge,\varphi),\qquad \varphi=\psi^\wedge,\forall\psi\in\widetilde{U},$$

which implies for these φ

$$(f^\wedge,v\varphi)=(v^{-1}g^\wedge,v\varphi).$$

Obviously $\{v\varphi(v):\varphi\in U\}$ is dense in $L^q_{R^+}$, so $f=v^{-1}g^\wedge$ in the distribution sense. Then by Lemma 7, it follows $f=I^{\langle 1\rangle}g$, thus $I^{\langle 1\rangle}g\in L^q_{R^+}$. In virtue of Theorem 3, $D^{\langle 1\rangle}(I^{\langle 1\rangle}g)=g$, i.e $D^{\langle 1\rangle}f=g$, which means $f\in W_1$.

For $r>1$, one may verify by nduction.

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逻辑导数与逻辑积分（Ⅱ）*

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摘　　要

在〔4〕中我们对空间$L^q_{R^+}$,$1\le q\le 2$,讨论了函数的逻辑导数与积分．例如，建立了下列公式：

$$D^{\langle 1\rangle}(I^{\langle 1\rangle}f)=f,\ I^{\langle 1\rangle}(D^{\langle 1\rangle}f)=f.$$

但那里的方法不能用于$q>2$情形．本文是〔4〕的继续．对$2<q<\infty$情形，我们利用分布理论与p进群的技巧定义空间$L^q_{R^+}$的Walsh-Fourier变式（WFT）并建立有关逻辑导数与积分的某些基本定理．