泛函微分方程的混合方法*

潘劲松
（广西大学）

1 引　　言

Gear、Gragg与Stetter最早提出了常微分方程的混合方法（见〔1〕、〔6〕），Makroglou把这一方法推广到了一类积分微分方程．本文将讨论Volterra泛函　分方程和中立型的泛函微分方程的混合方法，并给出了预测－校正的混合方法的收敛性证明和一些数值例子．

令$I=[t_0, t_0+a]$为一区间，记$C_n^0(I)$为具有范数$\|x\|^I=\sup\{\|x(s)\|: s\in I\}$从$I\to R^n$上的连续函数的空间，记$C_n^1(I)\subset C_n^0(I)$为有一阶连续导数的函数空间．考虑Volterra泛函微分方程组：

$$\begin{cases} y'(t)=f(t, y(\cdot)) & t\in I, \\ y(t)=\varphi(t) & t\in[t_0-r, t_0] \end{cases} \tag{1.1}$$

这里$\varphi\in C_n[t_0-r, t_0]$是指定的初值函数，$r\geq 0$是一个常数．其中算子$f$: $I\times C_n^0(I)\to R^n$满足下列条件：

A 1　对固定的$y\in C_n^0(I)$，映射$t\to f(t, y(\cdot))$在I上连续．

A 2　算子f满足Lipschitz条件：

$$\|f(t, y_1)-f(t, y_2)\|\leq L\|y_1-y_2\|^{[t_0, t]}$$

Driver（见〔7〕）证明了问题（1.1）的解的存在唯一性．

下面考虑中立型方程：

$$\begin{cases} y'(t)=f(t, y(\cdot), y'(\cdot)) & t\in I \\ y(t_0)=y_0 \end{cases} \tag{1.2}$$

其中f: $I\times C_n^1(I)\times C_n^0(I)\to R^n$，满足下列条件：

H 1　对固定$x\in C_n^1(I)$，映射$t\to f(t, x(\cdot), x'(\cdot))$在I上连续．

H 2　算子f满足Lipschitz条件：

$$\|f(t, x_1(\cdot), y_1(\cdot))-f(t, x_2(\cdot), y_2(\cdot))\|\leq L_1\|x_1-x_2\|^{[t_0, t]}+L_2\|y_1-y_2\|^{[t_0, t]}$$

其中$L_1\geq 0$，$0\leq L_2<1$为常数，$t\in I$, x_1, $x_2\in C_n^1(I)$, y_1, $y_2\in C_n^0(I)$．

则问题（1.2）存在唯一的解$y\in C_n^1(I)$（见〔8〕）

2　定义与记号

考虑一般混合方法：

$$y_h(t_{i+k-1}+rh)=\sum_{j=0}^{k-1}a_j(r)y_h(t_{i+j})+h\sum_{j=0}^{k}\beta_j(r)f_h(t_{i+j})+h\sum_{j=1}^{v}c_j(r)f_n(t_{i+k-1}+\theta_j h), \tag{2.1}$$

* 1983年7月19日收到．

其中$r\in[0,1]$，$0<\theta_j<1, j=1,2,\cdots,\nu$. 及一般线性多步方法：

$$\overline{y}_h(t_{i+k-1}+sh)=\sum_{j=0}^{k-1}\overline{a}_j(s)y_h(t_{i+j})+h\sum_{j=0}^{k-1}\overline{\beta}_j(s)f_h(t_{i+j}),\quad s\in[0,1]. \tag{2.2}$$

对于（2.1）式我们可定义：

$$\eta(t,r,h)=y(t+(k-1+r)h)-\sum_{j=0}^{k-1}a_j(r)y(t+jh)-h\sum_{j=0}^{k}\beta_j(r)y'(t+jh)$$
$$-h\sum_{j=1}^{\nu}c_j(r)y'(t+(k-1+\theta_j)h) \tag{2.3}$$

及 $\eta(h)=\sup\{\|\eta(t,r,h)\|:t\in[t_0,t_0+a-kh],\quad r\in[0,1]\}$

$\mu(h)=\sup\{\|\eta(t,1,h)\|:t\in[t_0,t_0+a-kh]\}$,

$\nu(h)=\sup\{\|\frac{\partial}{\partial r}\eta(t,r,h)\|:t\in[t_0,t_0+a-kh],\quad r\in[0,1]\}$.

同样对于（2.2）可定义：

$$\overline{\eta}(t,s,h)=\overline{y}(t+(k-1+s)h)-\sum_{j=0}^{k-1}\overline{a}_j(s)y(t+jh)-h\sum_{j=0}^{k-1}\overline{\beta}_j(s)y'(t+jh) \tag{2.4}$$

及 $\overline{\eta}(h)=\sup\{\|\overline{\eta}(t,s,h)\|:t\in[t_0,t_0+a-kh],\ s\in[0,1]\}$,

$\overline{\nu}(h)=\sup\{\|\frac{\partial}{\partial s}\overline{\eta}(t,s,h)\|:t\in[t_0,t_0+a-kh],\ s\in[0,1]\}$.

定义1（根条件）：如果多项式：

$$\rho(z)=a_k(1)z^k+a_{k-1}(1)z^{k-1}+\cdots+a_0(1)$$

的所有根均在单位园 $|z|=1$内或者单位园上只有单根，则称ρ满足根条件.

3 方法的构造与收敛性

3.1 在本节中，我们将讨论对Volterra型方程的混合方法及其收敛性. 对问题（1.1）予测－校正的混合方法可写为：

$$P_k:\ \overline{y}_h(t_{i+k})=\sum_{j=0}^{k-1}\overline{a}_j(1)y_h(t_{i+j})+h\sum_{j=0}^{k-1}\overline{\beta}_j(1)f_h(t_{i+j}),\overline{f}_h(t_{i+k})=f(t_{i+k},\overline{y}_h(\cdot)), \tag{3.3}$$

$$P_\nu:\ \overline{y}_h(t_{i+k-1}+sh)=\sum_{j=0}^{k-1}\overline{a}_j(s)y_h(t_{i+j})+h\sum_{j=0}^{k-1}\overline{\beta}_j(s)f_h(t_{i+j}),\ s\in[0,1]. \tag{3.4}$$

在实际计算中只要令s等于某个θ_j即可.

$$\overline{f}_h(t_{i+k-1}+\theta_jh)=f(t_{i+k-1}+\theta_jh,\ \overline{y}_h(\cdot)),\quad j=1,2\cdots,\nu$$

$$C:\ y_h(t_{i+k-1}+rh)=\sum_{j=0}^{k-1}a_j(r)y_h(t_{i+j})+h\sum_{j=0}^{k-1}\beta_j(r)f_h(t_{i+j})+h\beta_k(r)\overline{f}_h(t_{i+k})$$
$$+h\sum_{j=1}^{\nu}c_j(r)\overline{f}_h(t_{i+k-1}+\theta_jh),\ r\in[0,1],\quad 0<\theta_j<1, j=1,2,\cdots,\nu,$$
$$f_h(t_{i+k})=f(t_{i+k},y_h(\cdot)) \tag{3.5}$$

下面我们给出对问题（1.1）该混合方法的相容性定义.

定义2 如果$\mu(h)=o(h)$，$\eta(h)=o(1)$ 和$\overline{\eta}(h)=o(1)$，则称该混合方法是相容的. 如果$\mu(h)=O(h^{p+1})$，$\eta(h)=O(h^p)$，$\overline{\eta}(h)=O(h^p)$，则称该混合方法是$p$阶相容的.

又定义：$\varepsilon_h=y-y_h$，$d_h=f-f(t,y_h)$，$\overline{\varepsilon_h}=y-\overline{y}_h$，$\overline{d}_h=f(t,y)-f(t,\overline{y}_h)$

定理1 假定〈i〉多项式$\rho(z)=a_k(1)z^k+a_{k-1}(1)z^{k-1}+\cdots+a_0(1)$满足根条件；〈ii〉$y(t)$在I上充分连续可微；〈iii〉该混合方法是$p$阶相容的；
〈iv〉$\|\varepsilon_n\|^{[t_0,t_{k-1}]}\leq Qh^p$，$\|\bar{\varepsilon}_h\|^{[t_0,t_{k-1}]}\leq Qh^p$，$Q$为常数，及$h<h_0$，$h_0>0$为一常数．
则预测-校正的混合方法是收敛的，且收敛的阶为p．

证明 为了证明上的方便，我们令$\theta_0=1$及$c_0(r)=\beta_k(r)$，则（3.5）可以写为

$$y_h(t_{i+k-1}+rh)=\sum_{j=0}^{k-1}a_j(r)y_h(t_{i+j})+h\sum_{j=0}^{k}\beta_j(r)f_h(t_{i+j})+h\sum_{j=0}^{\nu}c_j(r)\bar{f}_h(t_{i+k-1}+\theta_jh)\tag{3.6}$$

令$r=1$，将上式与（2.3）相减得：

$$\varepsilon_h(t_{i+k})=\sum_{j=0}^{k-1}a_j(1)\varepsilon(t_{i+j})+h\sum_{j=0}^{k-1}\beta_j(1)d_h(t_{i+j})+h\sum_{j=0}^{\nu}c_j(1)\bar{d}_h(t_{i+k-1}+\theta_jh)+\eta(t_i,1,h)\tag{3.7}$$

注意条件〈i〉和〈ii〉，利用文献〔2〕、〔4〕、〔5〕上的方法可得：对$0\leq i\leq N$、c_1，c_2，c_3，c，$|\sigma|=\sum_{j=0}^{k-1}|\beta_j(1)|$，均为常数，有

$$\|\varepsilon_n(t_i)\|\leq c_1h^p+hc_2\Big[|\sigma|\sum_{j=0}^{i-1}\|d_h(t_j)\|+c\sum_{j=k-1}^{i-1}\sum_{j_1=0}^{\nu}\|\bar{d}_n(t_j+\theta_{j_1}h)\|\Big]+c_3\mu(h)/h.\tag{3.8}$$

因
$$\|\bar{d}_h(t)\|=\|f-\bar{f}_n\|\leq L\|y-y_h\|^{[t_0,t]},\tag{3.9}$$

且 $\bar{\varepsilon}_h(t_{i+k-1}+sh)=\sum_{j=0}^{k-1}\bar{a}_j(s)\varepsilon_h(t_{i+j})+\sum_{j=0}^{k-1}\bar{\beta}_j(s)d_h(t_{i+j})+\bar{\eta}(t_i,s,h)$，

取范数可得：

$$\|\bar{\varepsilon}_h(t_{i+k-1}+sh)\|\leq\bar{M}\sum_{j=0}^{k-1}\|\varepsilon_h(t_{i+j})\|+h\bar{N}\sum_{j=0}^{k-1}\|d_h(t_{i+j})\|+\bar{\eta}(h),$$

其中：$\bar{M}=\max\limits_{0\leq j\leq k-1}\{\sup\limits_{s\in[0,1]}|\bar{a}_j(s)|\}$，$\bar{N}=\max\limits_{0\leq j\leq k-1}\{\sup\limits_{s\in[0,1]}|\bar{\beta}_j(s)|\}$．

由上式对s的一致性可得：

$$\|\bar{\varepsilon}_h\|^{[t_{i+k-1},t_{i+k}]}\leq(\bar{M}+h\bar{N})\sum_{j=0}^{k-1}\|\varepsilon\|^{[t_0,t_{i+j}]}+\bar{\eta}(h)\tag{3.10}$$

因 $\|\bar{\varepsilon}_h\|^{[t_0,t_{k-1}]}\leq Qh^p$，则由于（3.10）的右端对$i$为非减序列，有

$$\|\bar{\varepsilon}_h\|^{[t_0,t_{i+k}]}\leq Qh^p+\bar{u}\sum_{j=0}^{k-1}\|\varepsilon_h\|^{[t_0,t_{i+j}]}+\bar{\eta}(h),$$

其中$\bar{u}=\bar{M}+h_0\bar{N}L$上式显然可改写为；

$$\|\bar{\varepsilon}_h\|^{[t_0,t_j]}\leq Qh^p+\bar{u}\sum_{j=0}^{k-1}\|\varepsilon_h\|^{[t_0,t_{i-j-k}]}+\bar{\eta}(h).\tag{3.11}$$

将（3.9）、（3.11）代入（3.8）得：

$$\|\varepsilon(t_i)\|\leq c_1h^p+hc_2\Big[c\sum_{j=k-1}^{i-1}(\nu+1)L\|\bar{\varepsilon}_h\|^{[t_0,t_{j-1}]}+c(\nu+1)\sigma LQh^{p-1}+|\sigma|\sum_{j=0}^{i-1}L\|\varepsilon\|^{[t_0,t_j]}\Big]+c_3\mu(h)/h\leq c_1'h^p+$$

$$+hc_2\Big[(v+1)cL\sum_{j=k-1}^{i-1}\big(\bar{u}\sum_{j_1=0}^{k-1}\|\varepsilon_h\|^{[t_0,t_{j-j_1-k}]}+\bar{\eta}(h)\big)+|\sigma|L\sum_{j=0}^{i-1}\|\varepsilon_h\|^{[t_0,t_j]}\Big]+c_3\mu(h)/h$$

$$<c_1'h^p+hc_2\Big[(v+1)cL\big(k\bar{u}\sum_{j=0}^{i-1}\|\varepsilon_h\|^{[t_0,t_j]}+\frac{a}{h}\bar{\eta}(h)\big)+|\sigma|L\sum_{j=0}^{i-1}\|\varepsilon_h\|^{[t_0,t_j]}\Big]+$$

$$+c_3\mu(h)/h$$

故 $$\|\varepsilon_h(t_i)\|\le c_1'h^p+hu\sum_{j=0}^{i-1}\|\varepsilon_h\|^{[t_0,t_j]}+u_1\bar{\eta}(h)+c_3\mu(h)/h \tag{3.12}$$

其中 $$u=Lc_2[c(v+1)L\bar{u}K+|\sigma|]$$

$$u_1=c_2(v+1)cLa,\qquad c_1'=c_1+c_2c(v+1)aLQ$$

考虑下式：

$$\varepsilon_h(t_{i+k-1}+rh)=\sum_{j=0}^{k-1}a_j(r)\varepsilon_h(t_{i+j})+h\sum_{j=0}^{k-1}\beta_j(r)d_h(t_{i+j})$$

$$+h\sum_{j=0}^{v}c_j(r)\bar{d}_h(t_{i+k-1}+\theta_jh)+\eta(t_i,r,h)$$

由于上式对r是一致成立的：

$$\|\varepsilon_h\|^{[t_{i+k-1},t_{i+k}]}\le M'\sum_{j=0}^{k-1}\|\varepsilon_h(t_{i+j})\|+hN'\sum_{j=0}^{k-1}\|d_h(t_{i+j})\|$$

$$+hc'\sum_{j=0}^{v}\|\bar{d}_h(t_{i+k-1}+\theta_jh)\|+\eta(h)$$

其中 $M'=\max\limits_{0\le j\le k-1}\{\sup\limits_{r\in[0,1]}|a_j(r)|\}$，$N'=\max\limits_{0\le j\le k-1}\{\sup\limits_{r\in[0,1]}|\beta_j(r)|\}$，

$$c'=\max_{0\le j\le v}\{\sup_{r\in[0,1]}|c_j(r)|\}$$

将（3.12）代入上式：

$$\|\varepsilon_h\|^{[t_{i+k-1},t_{i+k}]}\le M'\sum_{j=0}^{k-1}\Big[c_1h^p+hu\sum_{j_1=0}^{i+j-1}\|\varepsilon_h\|^{[t_0,t_{j_1}]}+u_1\bar{\eta}(h)+$$

$$c_3\mu(h)/h\Big]+hc'L(v+1)\|\varepsilon_h\|^{[t_0,t_{i+k}]}+LN'h\sum_{j=i}^{i+k-1}\|\varepsilon_h\|^{[t_0,t_j]}+\eta(h)\le M'Kc_1h^p+$$

$$M'Kuh\sum_{j=0}^{i+k-2}\|\varepsilon_h\|^{[t_0,t_j]}+hc'L(v+1)\Big[Qh^p+\bar{u}\sum_{j=0}^{k-1}\|\varepsilon_h\|^{[t_0,t_{i+j}]}+\bar{\eta}(h)\Big]+$$

$$LN'h\sum_{j=i}^{i+k-1}\|\varepsilon_h\|^{[t_0,t_j]}+\eta(h)+u_1M'K\bar{\eta}(h)+c_3M'K\mu(h)/h\le(M'Ku+c'L(v+1)\bar{u}+$$

$$LN')h\sum_{j=0}^{i+k-1}\|\varepsilon_h\|^{[t_0,t_j]}+D^*h^p+(c'L(v+1)h+u_1M'K)\bar{\eta}(h)+c_3M'K\mu(h)/h+\eta(h),$$

其中$D^*=\max(Q,M'Kc_1+Qc'L(v+1)h_0)$．由上式右端之非减性，知

$$\|\varepsilon_h\|^{[t_0,t_{i+k}]}\le A_1\sum_{j=0}^{i+k-1}\|\varepsilon_h\|^{[t_0,t_j]}+A_2(h), \tag{3.13}$$

其中：$A_1=M'Ku+c'L\bar{u}(v+1)+LN'$，$A_2(h)=D^*h^p+[c'L(v+1)h_0+u_1M'K]\bar{\eta}(h)+c_3M'K\mu(h)/h+\eta(h)$．

令z_i等于（3.13）式的右端，则

$$\|\varepsilon_h\|^{[t_0,\ t_{i+k}]}\le z_i;\quad i=0,1,\cdots,N-k,$$

$$z_{i+1}-z_i\le hA_1\|\varepsilon_h\|^{[t_0,\ t_{i+k}]}=hA_1z_i$$

故对$i=0,1,\cdots;N-k-1$，从不等式$z_{i+1}-z_i\le hA_1z_i$可得$z_{i+1}\le(1+hA_1)z_i$。从而$z_i\le(1+hA_1)^iz_0\le\exp(A_1(t_i-t_0))z_0$。并且$\|\varepsilon_h\|^1\le A_2\exp(A_1a)$，$a$为区间$I$的长度。

故定理得证，予测－校正的混合方法是收敛的，且收敛的阶为p。

3.2 在本节中，我们将讨论中立型方程的混合方法及其收敛性。对问题（1.2）予测－校正的混合方法可描述如下：

$$\mathrm{P}_k:\quad \bar{y}_h(t_{i+k})=\sum_{j=0}^{k-1}\bar{a}_j(1)y_h(t_{i+j})+h\sum_{j=0}^{k-1}\bar{\beta}_j(1)z_h(t_{i+j}),$$

$$\bar{z}_h(t_i+sh)=\bar{y}'_h(t_i+sh),\quad s\in(0,1),$$

$$\bar{z}_h(t_{i+k})=f(t_{i+k},\ \bar{y}_h(\cdot),\ \bar{z}_h(\cdot));\tag{3.14}$$

$$\mathrm{P}_v:\quad \bar{y}_h(t_{i+k-1}+sh)=\sum_{j=0}^{k-1}\bar{a}_j(s)y_h(t_{i+j})+h\sum_{j=0}^{k-1}\bar{\beta}_j(s)z_h(t_{i+j}),\quad s\in(0,1).\tag{3.15}$$

在实际计算中，只要取s等于某个θ_j即可。现令$t_{i+k-1}+\theta_jh=t^*_{q+k}$，$j=1,2,\cdots,v$，则

$$\bar{z}_h(t^*_q+sh)=\bar{y}'_h(t^*_q+sh),\quad s\in(0,1),$$

$$\bar{z}_h(t^*_{q+k})=f(t^*_{q+k},\ \bar{y}_n(\cdot),\ \bar{z}_n(\cdot));$$

$$\mathrm{C}:\quad y_h(t_{i+k-1}+rh)=\sum_{j=0}^{k-1}a_j(r)y_h(t_{i+j})+h\sum_{j=0}^{k-1}\beta_j(r)z_h(t_{i+j})+h\beta_k(r)\bar{z}_h(t_{i+k})+$$

$$+h\sum_{j=1}^{v}c_j(r)\bar{z}_h(t_{i+k-1}+\theta_jh),\tag{3.16}$$

$$r\ [0,1],\ 0<\theta_j<1,\quad 0<j=1,2,\cdots,v.$$

$$z_h(t_i+rh)=y'_h(t_i+rh),\quad r\in(0,1),$$

$$z_h(t_{i+k})=f(t_{i+k},y_h(\cdot),\ z_h(\cdot)).$$

下面我们给出对中立型方程混合方法的相容性定义.

定义 3 如果$\mu(h)=o(h)$，$\eta(h)=o(1)$，$\bar{\eta}(h)=o(1)$，$v(h)=o(h)$，$\bar{v}(h)=o(h)$,则称该混合方法是相容的。如果$\mu(h)=O(h^{p+1})$，$\eta(h)=O(h^p)$，$\bar{\eta}(h)=O(h^p)$，$v(h)=O(h^{p-1})$ $\bar{v}(h)=O(h^{p-1})$，则称该混合方法是p阶相容的。

定理 2 假定

〈i〉算子 $\rho(z)=a_k(1)z^k+a_{k-1}(1)z^{k-1}+\cdots+a_0(1)$

满足根条件；

〈ii〉混合方法是p阶相容的；〈iii〉$y(t)$ 在I上充分连续可微；

〈iv〉 $\|\varepsilon_n\|^{[t_0,\ t_{k-1}]}\le Qh^p$，$\|\bar{\varepsilon}_h\|^{[t_0,t_{k-1}]}\le Qh^p$，$\|d_h\|^{[t_0,t_{k-1}]}\le Q_1h^p$，$\|\bar{d}_h\|^{[t_0,t_{k-1}]}\le Q_1h^p$.

Q_1、Q_2均为常数，$h<h_0$，$h_0>0$为常数。则予测－校正的混合方法是收敛的。

证明 同定理1一样，我们有：

$$\|\varepsilon_h(t_i)\|\le c_1h^p+hc_2\Big[c\sum_{j=k-1}^{i-1}\sum_{j_1=1}^{v}\|d_h(t_j+\theta_jh)\|+|\sigma|\sum_{j=0}^{i-1}\|d_h(t_j)\|\Big]+c_3\mu(h)/h\tag{3.17}$$

令$t^*=t_{i-1}+rh$，$r\in[0,1]$，有

$\|d_h(t^*)\| \le \|f(t^*, y(\cdot), z(\cdot)) - f(t^*, y_h(\cdot), z_h(\cdot))\| + \|f(t^*, y_h(\cdot), z_h(\cdot)) - z_h(t^*)\| \le L_1 \|\varepsilon_h\|^{[t_0, t^*]} + L_2 \|d_h\|^{[t_0, t^*]} + v(h)/h \le L_1 \|\varepsilon_h\|^{[t_0, t_i]} + L_2 \|d_h\|^{[t_0, t_i]} + v(h)/h$.

因 $\|d_h\|^{[t_0, t_{i-1}]} \le Q_1 h^p$，且由于上式对$t$是一致成立的，故

$$\|d_h\|^{[t_{i-1}, t_i]} \le Q_1 h^p + L_1 \|\varepsilon_h\|^{[t_0, t_i]} + L_2 \|d_h\|^{[t_0, t_i]} + v(h)/h$$

由上式右端之非减性：

$$\|d_h\|^{[t_0, t_i]} \le Q_1 h^p + L_1 \|\varepsilon_h\|^{[t_0, t_i]} + L_2 \|d_h\|^{[t_0, t_i]} + v(h)/h$$

故
$$\|d_h\|^{[t_0, t_i]} \le Q_2 h^p + B \|\varepsilon_h\|^{[t_0, t_i]} + B_1 v(h)/h, \tag{3.18}$$

其中：
$$Q_2 = \frac{Q_1}{1 - L_2}, \quad B = \frac{L_1}{1 - L_2}, \quad B_1 = \frac{1}{1 - L_2}.$$

同理，因 $\|\overline{d}_h\|^{[t_0, t_{k-1}]} \le Q_1 h^p$，故

$$\|\overline{d}_h\|^{[t_0, t_i]} \le Q_2 h^p + \overline{B} \|\overline{\varepsilon}_h\|^{[t_0, t_i]} + \overline{B}_1 \overline{v}(h)/h, \tag{3.19}$$

其中Q_2、$\overline{B}$、$\overline{B}_1$均为常数．对（3.15）式取范数得：

$$\|\overline{\varepsilon}_h(t_{i-k-1} + sh)\| \le \overline{M} \sum_{j=0}^{k-1} \|\varepsilon_h(t_{i-j})\| + h\overline{N} \sum_{j=0}^{k-1} \|d_h(t_{i-j})\| + \overline{\eta}(h),$$

其中$\overline{M}$、$\overline{N}$与定理1中所定义的一样．又由上式对s的一致性；有

$$\|\overline{\varepsilon}_h\|^{[t_{i-k-1}, t_{i-k}]} \le \overline{M}K \|\varepsilon_h\|^{[t_0, t_{i-k-1}]} + h\overline{N}K \|d_h\|^{[t_0, t_{i-k-1}]} + \overline{\eta}(h)$$

将（3.18）代入上式得：

$$\begin{aligned}\|\overline{\varepsilon}_h\|^{[t_{i-k-1}, t_{i-k}]} &\le \overline{M}K \|\varepsilon_h\|^{[t_0, t_{i-k-1}]} + h\overline{N}K(Q_2 h^p + B \|\varepsilon_h\|^{[t_0, t_{i-k-1}]} + B_1 v(h)/h) + \overline{\eta}(h) \\ &\le D^* h^p + (\overline{M}K + h\overline{N}KB) \|\varepsilon_h\|^{[t_0, t_{i-k-1}]} + K\overline{N}B_1 v(h)/h + \overline{\eta}(h),\end{aligned}$$

其中$D^* = \max\{Q, K\overline{N}Q_2 h_0\}$，由上式右端之非减性可得：

$$\|\overline{\varepsilon}_h\|^{[t_0, t_{i-k}]} \le D^* h^p + \overline{u} \|\varepsilon_h\|^{[t_0, t_{i-k-1}]} + \overline{D}v(h) + \overline{\eta}(h), \tag{3.20}$$

其中$\overline{u} = \overline{M}K + h_0 \overline{N}KB$，$D = K\overline{N}B_1$．将（3.20）代入（3.19）可得：

$$\begin{aligned}\|\overline{d}_h\|^{[t_0, t_j]} &\le Q_2 h^p + \overline{B}(D^* h^p + \overline{u} \|\varepsilon_h\|^{[t_0, t_{j-1}]} + \overline{D}v(h) + \overline{\eta}(h)) + \overline{B}_1 \overline{v}(h)/h \\ &\le E_1 h^p + E_2 \|\varepsilon_h\|^{[t_0, t_{j-1}]} + E_3 v(h)/h + E_4 \overline{\eta}(h) + E_5 \overline{v}(h)/h,\end{aligned} \tag{3.21}$$

$E_1 = Q_2 + \overline{B}D^*$，$E_2 = \overline{B}\overline{u}$，$E_3 = \overline{B}\,\overline{D}h_0$，$E_4 = \overline{B}$，$E_5 = \overline{B}_1$．

将（3.18）、(3.21）代入（3.17）有：

$$\begin{aligned}\|\varepsilon_h(t_i)\| \le c_1 h^p + hc_2\Big[c(v+1) \sum_{j=k-1}^{i-1} (E_1 h^p + E_2 \|\varepsilon_h\|^{[t_0, t_j]} + E_3 v(h)/h + E_4 \overline{\eta}(h) \\ + E_5 \overline{v}(h)/h) + |\sigma| \sum_{j=0}^{i-1} (Q_2 h^p + B \|\varepsilon_h\|^{[t_0, t_j]} + B_1 \mu(h)/h)\Big] + c_3 v(h)/h,\end{aligned}$$

故
$$\|\varepsilon_h(t_i)\| \le u_1 h^p + hu \sum_{j=0}^{i-1} \|\varepsilon_h\|^{[t_0, t_j]} + G(h), \tag{3.22}$$

$u_1 = c_1 + c_2 a[c(v+1)E_1 + Q_2 |\sigma|]$，　$u = c_2[c(v+1)E_2 + B|\sigma|]$，

$G(h) = ac_2[c(v+1)E_3 v(h)/h + c(v+1)E_5 \overline{v}(h)/h + c(v+1)E_4 \eta(h) + |\sigma| B_1 v(h)/h] + c_3 \mu(h)/h$.

显然　$G(h) = O(h^p)$．考虑下式：

$$\varepsilon_h(t_{i-k-1} + rh) = \sum_{j=0}^{k-1} \alpha_j(r)\varepsilon_h(t_{i+j}) + h \sum_{j=0}^{k-1} \beta_j(r) d_h(t_{i+j}) + h \sum_{j=0}^{v} c_j(r) \overline{d}_h(t_{i+k-1} + \theta_j h) +$$

$\eta(t_i, r, h)$.

对上式取范数，并由上式对r的一致性：

$$\|\varepsilon_h\|^{[t_{i+k-1}, t_{i+k}]} \le M' \sum_{j=0}^{k-1} \|\varepsilon_h(t_{i+j})\| + hN' \sum_{j=0}^{k-1} \|d_h(t_{i+j})\|$$

$$+ hc' \sum_{j=0}^{v} \|\overline{d}_h(t_{i+k-1} + \theta_j h)\| + \eta(h),$$

其中M'，N'，c'与定理1中的一样为常数，

故 $\|\varepsilon_h\|^{[t_{i+k-1}, t_{i+k}]} \le M' \sum_{j=0}^{k-1} \Big(u_1 h^p + hu \sum_{j_1=0}^{i+j-1} \|\varepsilon_h\|^{[t_0, t_j]} + G(h) \Big) - hN' \sum_{j=0}^{k-1} (Q_2 h^p +$

$B\|\varepsilon_h\|^{[t_0, t_{i+j}]} + B_1 v(h)/h) + hc' \sum_{j=0}^{v} (E_1 h^p + E_2 \|\varepsilon_h\|^{[t_0, t_{i+k-1}]} + E_3 v(h)/h + E_4 \overline{\eta}(h) +$

$E_5 \overline{v}(h)/h) + \eta(h)$,

整理得：

$$\|\varepsilon_h\|^{[t_{i+k-1}, t_{i+k}]} \le D^{**} h^p + hA \sum_{j=0}^{i+k-1} \|\varepsilon_h\|^{[t_0, t_j]} + G_1(h), \tag{3.23}$$

其中：$D' = M' K u_1 + (N' K Q_2 + c'(v+1)) h_0$,

$$D^{**} = \max(Q, D'), \quad A = M' K u + BN' + c' E_2 (v+1),$$

$G_1(h) = KM' G(h) + N' K B_1 v(h) + E_3 c'(v+1) v(h) + hc'(v+1) E_4 \overline{\eta}(h) + \eta(h) +$
$+ E_5 c'(v+1) \overline{v}(h)$.

显然$G_1(h) = O(h^p)$。因$D^{**} \ge Q$，故由（3.22）式的非减性知

$$\|\varepsilon_h\|^{[t_0, t_{i+k}]} \le D^{**} h^p + hA \sum_{j=0}^{i+k-1} \|\varepsilon_h\|^{[t_0, t_j]} + G_1(h) \le hA_1 \sum_{j=0}^{i+k-1} \|\varepsilon_h\|^{[t_0, t_j]} + A_2(h),$$

其中　$A_1 = A$，$A_2(h) = D^{**} h^p + G_1(h)$。

仿照定理1，由上一不等式可推得：

$$\|\varepsilon_h\|^I \le A_2(h) \exp(A_2 a).$$

从而该混合方法对问题（1.2）是收敛的，且收敛的阶为p。

4　数值例子

下面我们给出一组预测－校正公式和一个数值例子：

$$\mathrm{P}_k: \quad \overline{y}_h(t_{i+1}) = -9y_h(t_i) + 9y_h(t_{i-1}) - y_h(t_{i-2}) + h[6z_h(t_i) + 6z_h(t_{i-1})], \tag{4.1}$$

$$\mathrm{P}_v: \quad \overline{y}_h(t_i + h/2) = \{-45y_h(t_i) + 100y_h(t_{i-1}) + 9y_h(t_{i-2}) + h[90z_h(t_i) + 60z_h(t_i)]\}/64, \tag{4.2}$$

$$\mathrm{C}: \quad y_h(t_i + rh) = y_h(t_i) + h[(R + 2R^3/3 - 3R^2/2) z_h(t_i) + (2R^3/3 - R^2/2) \overline{z}_h(t_{i+1}) + (2R^2 - 4R^3/3) \overline{z}_h(t_i + H/2)] \tag{4.3}$$

例　$y'(t) = \cos(t)(1 + u(t)) + y(t)z(t) - \sin(t(1 + \sin^2(t)))$,

$$0 \le t \le 1, \quad y(0) = 0, \quad y'(0) = 1.$$

其中 $u(t) = y(ty^2(t))$，$z(t) = y'(ty^2(t))$。精确解为：$y(t) = \sin(t)$。

利用公式（4.1)(4.2)(4.3）算得这一例子的数值结果如下表所示：

t	$y(t)$	$y_h(2^{-2})$	$y_h(2^{-4})$	$y_h(2^{-6})$	$y_h(2^{-8})$
0	0.	0.	0.	0.	0.
0.25	0.2474039593	0.2474039593	0.2474039569	0.2474039593	0.2474039593
0.50	0.4794255386	0.4794255386	0.4794255300	0.4794255386	0.4794255386
0.75	0.6816387600	0.6816311222	0.6816387404	0.6816387599	0.6816387600
1.00	0.8414709848	0.8414578438	0.8414709562	0.8414709845	0.8414709848

本文是在苏德富老师的指导下完成的，谨致谢意.

参 考 文 献

[1] Gear, C.W., Hybrid methods for initial value problems in ordinary differential equations, SINUN 2, 69—86, 1964.

[2] Makroglou, A., Hybrid methods in the numerical solution of Volterra integro differential equations, IMA J. Nomer. Anal., 2, 21—35, 1982.

[3] Jackiewicz, ., The numerical solution of Volterra functional differential equation of neutral type, SIMA J. Numer. Anal., Vol. 18, No. 4, August, 1981.

[4] Tavernini, L., Linear multistep methods for the numerical solution of Volterra functional differential equation, Applicable Anal. 1 (1973), pp169—185.

[5] Henrici, P., Discrete Variable Methods in Ordinary Differential Equations, New York, Joho Wiley, 1962, pp242—244.

[6] Gragg, W.B. and Stetter, H.J., Generalized multistep predictor corrector methods, J. Ass. Comput. Mach. 11 (1964), pp188—209.

[7] Driver, R.D., Existence and stability of solutions of a delay-differential system, Arch. Rational Mach. Anal., 10 (1962), 401—406.

[8] Komont, Z. and Kwapisz, On the Cauchy problem for differential delay equations in a Banach space, Math. Nachr., 7 (1976), pp. 173—190.

Hybrid Methods in Functional Differential Equations

Pan Jing song (Guangxi University)

Abstract

In 1964, Gear, Gragg and stetter discovered a hubrid method in ordinary differential equation. Makroqlou extended this method to a sort of integro-differential equation. I extend these results further and discuss hubrid methods in Volterra functional differential equation and functional differential equation of neutral type. I develope the hybrid methods of predictor corrector and give the proof of covergence and some numerical examples.