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A Comment on a Necessary and Sufficient Condition for Transitive Permutation Group by P. J. Cameron*

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P. J. Cameron had mentioned that "It can be shown that a permutation group is transitive if and only if its centralizer in the symmetric group is semiregular, and vice versa (Wielandt[68], page 9)."[1] The latter is true, *i.e.* $G \leqslant S_\Omega$ is semiregular $\Leftrightarrow C_{S_\Omega}(G)$ is transitive. [*cf.* 2] But in the former statement fails, *i.e.* $C_{S_\Omega}(G)$ is semiregular $\not\Rightarrow G \leqslant S_\Omega$ is transitive. In this note we give a series of the counterexample so that we illustrate that the condition above-mentioned isn't true.

We mention some examples.

Ex. 1 Set $\Omega = \{1, 2, 3, 4, 5, 6\}$.

Let $G = \langle a, b \rangle$, where $a = (123)(456)$, $b = (23)(56)$. By calculating, we get $a^3 = a^2 = 1$, $ba = a^2 b$. Hence every element of G can be uniquely expressed as $a^\lambda b^\mu$, where $\lambda = 0, 1, 2, 3$; $\mu = 0, 1$.

Let $\sigma = (14)(25)(36)$. It is easy to verify that $g^\sigma = g$, $\forall g \in G$ (In fact, because $a^\sigma = [(123)(456)]^{(14)(25)(36)} = (456)(123) = a$, $b^\sigma = [(23)(56)]^{(14)(25)(36)} = (56)(23) = b$, hence $g^\sigma = (a^\lambda b^\mu)^\sigma = (a^\lambda)^\sigma (b^\mu)^\sigma = (a^\sigma)^\lambda (b^\sigma)^\mu = a^\lambda b^\mu = g$, $\forall \lambda = 0, 1, 2, 3; \mu = 0, 1$.) Hence $\sigma \in C_{S_6}(G) \doteq \mathscr{C}$. Thus $\langle \sigma \rangle \leqslant \mathscr{C}$.

Again since $|C_{S_6}(a)| = 2!\,3^2 = 18$, $|C_{S_6}(b)| = 2!\,2^2 \cdot 2!\,1^2 = 16$ and $\mathscr{C} \leqslant C_{S_6}(a) \cap C_{S_6}(b)$, hence $|\mathscr{C}| \leqslant 2$. Obviously, $|\langle \sigma \rangle| = 2$, and so $\mathscr{C} \leqslant \langle \sigma \rangle$. Thus $\mathscr{C} = \langle \sigma \rangle$.

For any $a \in \Omega$, we have $a^{\sigma\sigma} = a^{\sigma^2} = a$, hence $\sigma^2 \in \mathscr{C}_a$. Therefore $\mathscr{C}_a = 1$, $\forall a \in \Omega$. It implies that $C_{S_6}(G)$ is semiregular.

It is easy to see that G is transitive on $\Omega_1 = \{1, 2, 3\}$ and $\Omega_2 = \{4, 5, 6\}$ respectively. Therefore G on Ω isn't transitive.

Ex. 1 can be generalized. More generally, we have

Ex. 2 Set $\Omega = \{1, 2, \cdots, m\}$, where $m = 2(2n+1)$, n is natural number. When $n = 1$, there is the special case of Ex. 1.

Let $G = \langle a, b \rangle$, where $a = \left(1, 2, \cdots, \frac{m}{2}\right)\left(\frac{m}{2}+1, \frac{m}{2}+2, \cdots, m\right)$, $b = \left(2, 3, \cdots, \frac{m}{2}\right)\left(\frac{m}{2}+2, \frac{m}{2}+3, \cdots, m\right)$, and inductively, by calculating, we get that every element of G can be uniquely expressed as $a^\lambda b^\mu$, where $\lambda = 0, 1, \cdots, \frac{m}{2}$; $\mu = 0, 1, \cdots$

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$\cdots, \frac{m}{2}-2$.

For $\sigma=(1, \frac{m}{2}+1)(2, \frac{m}{2}+2)\cdots(\frac{m}{2}, m)$. Since $a^\sigma=[(1\cdots\frac{m}{2})(\frac{m}{2}+1\cdots m)]^\sigma=$

$=(1^\sigma 2^\sigma\cdots(\frac{m}{2})^\sigma)((\frac{m}{2}+1)^\sigma(\frac{m}{2}+2)^\sigma\cdots m^\sigma)=(\frac{m}{2}+1, \frac{m}{2}+2, \cdots m)(1, 2, \cdots\frac{m}{2})=a$,

$b^\sigma=[(2, 3, \cdots, \frac{m}{2})(\frac{m}{2}+2, \cdots m)]^\sigma=(2^\sigma 3^\sigma\cdots(\frac{m}{2})^\sigma)((\frac{m}{2}+2)^\sigma\cdots m^\sigma)=(\frac{m}{2}+2, \frac{m}{2}+3, \cdots m)\cdot$

$(2, 3, \cdots, \frac{m}{2})=b$, hence $g^\sigma=(a^\lambda b^\mu)^\sigma=(a^\lambda)^\sigma(b^\mu)^\sigma=(a^\sigma)^\lambda(b^\sigma)^\mu=a^\lambda b^\mu=g$, $\forall g\in G$. Thus $\sigma\in\mathscr{C}=C_{S_\Omega}(G)$, *i.e.* $\langle\sigma\rangle\leqslant\mathscr{C}$.

Again since

$$|C_{S_\Omega}(a)|=2!\,(\frac{m}{2})^2=2!\,(\frac{2(2n+1)}{2})^2=2(2n+1)^2$$

$$|C_{S_\Omega}(b)|=2!\,(\frac{m-2}{2})^2\cdot 2!\,1^2=(m-2)^2=4(2n)^2$$

and $\mathscr{C}\leqslant C_{S_\Omega}(a)\cap C_{S_\Omega}(b)$, hence $|\mathscr{C}|\leqslant 2$ as $(2n, 2n+1)=1$. Obviously, $|\langle\sigma\rangle|=2$. If follows that $\mathscr{C}=\langle\sigma\rangle$. Thus $\mathscr{C}$ also is a cyclic group of order 2.

For any $\alpha\in\Omega$, $\alpha^{\sigma\sigma}=\alpha^{\sigma^2}=\alpha$, and so $\sigma^2\in\mathscr{C}_\alpha$. Hence $\mathscr{C}_\alpha=1$. This implies that $\mathscr{C}$ is semiregular. Obviously, G on Ω isn't transitive.

Remark In the Ex.2, if $m=2\cdot 2n$ ($n\geqslant 2$ is natural number), for instance, say $m=8$, *i.e* $\Omega=\{1, 2, \cdots, 8\}$. Let $G=\langle a, b\rangle$, where $a=(1234)(5678)$, $b=(234)(678)$. Then for $\sigma=(15)(26)(37)(48)$. Obviously, $\langle\sigma\rangle$ is also a cyclic group of order 2. But since $|C_{S_8}(a)|=2!\,4^2=32$, $|C_{S_8}(b)|=2!\,3^2\cdot 2!\,1^2=36$, and $\mathscr{C}=C_{S_8}(G)\leqslant C_{S_8}(a)\cap C_{S_8}(b)$, hence $|\mathscr{C}|\leqslant 4$. Therefore $\mathscr{C}\neq\langle\sigma\rangle$. This implies that Ex.1 can not be generalize in this case.

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