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On Uniform Convergence of the Riesz Means of Fourier Series on Compact Lie Groups*

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Let G be a compact connected Lie group and L be its Lie algebra. Let T be a maximal toral subgroup of G and H be the Lie algebra of T. Any f in L(G) has a Fourier series

$$f \sim \sum_{\lambda \in \Lambda(G)} d_\lambda \chi_\lambda * f(g) \tag{1}$$

Denote by $S_R^\delta(f;g)$ the Riesz means of order δ of (1),

$$S_R^\delta(f;g) = \sum_{\substack{\lambda \in \Lambda(G) \\ |\lambda+\beta| \le R}} \left(1 - \frac{|\lambda+\beta|^2}{R^2}\right)^\delta d_\lambda \chi_\lambda * f(g).$$

We call $\delta = \frac{n-1}{2}$ the critical index, where n is the dimension of G, and denote $S_R^{\frac{n-1}{2}}(f;g)$ by $S_R(f;g)$ simply.

Let $B(\cdot,\cdot)$ be the invariant inner product on L, and ρ be an invariant Riemannian metric associated with $B(\cdot,\cdot)$, so that for small t, $\rho(e,e^X) = |t||X|$, $X \in L$. Let $\gamma_0 = \sup\{\rho(g_1,g_2);\ g_1, g_2 \in G\}$. For $g_0 \in G$ we denote by $f_{g_0}(t)$ the spherical mean of the function f over a sphere $S(g_0,t)$ in G,

$$f_{g_0}(t) = |S(g_0,t)|^{-1} \int_{S(g_0,t)} f(g)\,d\sigma(g), \quad 0 < t \le \gamma_0$$

$$f_{g_0}(0) = f(g_0),$$

where $S(g_0,t) = \{g \in G;\ \rho(g,g_0) = t\}$ and $|S(g_0,t)| = \int_{S(g_0,t)} d\sigma(g)$ is the Haar measure of $S(g_0,t)$.

For f in C(G) we define some modulus of continuity of f as follows:

$$\omega(f;t) = \sup_{g_1, g_2 \in G} \sup_{\rho(g_2,e) \le t} |f(g_1 g_2) - f(g_1)|,$$

where e is the unit element of G,

$$\tilde{\omega}(f;t) = \sup_{g \in G} \sup_{0 \le \sigma \le t} \sup_{\tau \ge 0} |f_g(\tau+\sigma) - f_g(\tau)|$$

and

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$$\tilde{\omega}^{\Delta}(f;t)=\begin{cases} -I(t), & I(t)<0, \\ 0, & I(t)\geq 0, \end{cases}$$

where

$$I(t)=\inf_{g\in G}\ \inf_{0\leq\sigma\leq t}\ \inf_{\tau\geq 0}[f_g(\tau+\sigma)-f_g(\tau)].$$

$\tilde{\omega}^{\Delta}(f;t)$ is called the one-side modulus of continuity of f.

Let $\tilde{H}_{\omega}$ denote the set of all functions f such that $f\in C(G)$ and $\tilde{\omega}(f;t)=O(\omega(t))$, and $\tilde{H}^{\Delta}_{\omega}$ the set of all functions f such that $f\in C(G)$ and $\tilde{\omega}^{\Delta}(f;t)=O(\omega(t))$, where ω is a modulus of continuity. It's obvious that $\tilde{H}_{\omega}\subset\tilde{H}^{\Delta}_{\omega}$.

In this paper, we consider the uniform convergece of the Riesz means (the critical index) of Fourier series on G, and establish the following theorems, which generalize B. I. Golubov's result and T. I. Akhobadze's result on uniform convergence of multiple Fourier series (see [1], [2]).

Theorem 1 (i) Let $f\in C(G)$ such that

$$\tilde{\omega}(f;t)=o(1/\log\frac{1}{t}),\qquad t\to+0,$$

then $S_R(f;g)$ converges uniformly on G.

(ii) If ω is a modulus of continuity, satisfying the condition

$$\varlimsup_{t\to+0}\omega(t)\log\frac{1}{t}>0 \tag{2}$$

then there exists a function $f^{*}\in\tilde{H}_{\omega}$ such that $S_R(f^{*};e)$ is divergent.

Theorem 2 (i) Let $f\in C(G)$ such that

$$\tilde{\omega}^{\Delta}(f;t)=o(1/\log\frac{1}{t})\qquad t\to+0,$$

then $S_R(f;g)$ converges uniformly on G.

(ii) If ω is a modulus of continuity and satisfies (2), then there exists a function $f^{*}\in\tilde{H}^{\Delta}_{\omega}$ such that $S_R(f^{*};e)$ diverges.

To prove these theorems we need the following lemmas.

Let

$$\Delta(h)=\prod_{j=1}^{m}2i\sin[\frac{1}{2}B(h,a_j)],\quad D(h)=\prod_{j=1}^{m}B(h,a_j),\ h\in H,$$

where $a_1,\cdots,a_m$ are all positive roots on H.

Lemma 1 Let $F(h)=\Delta(h)-D(ih)$, $t=|h|\equiv[B(h,h)]^{\frac{1}{2}}$, $\eta=|h|^{-1}h$. Then

(i) $|F(t\eta)|\leq Ct^{m+2}$, $t\leq 1$,

(ii) $|F(t\eta)-F((t-\frac{\pi}{R})\eta)|\leq CR^{-1}t^{m}$, $\frac{\pi}{R}\leq t\leq 1$,

(iii) $|\Delta(t\eta)-\Delta((t-\frac{\pi}{R})\eta)|\leq CR^{-1}t^{m-1}$, $\frac{\pi}{R}\leq t\leq 1$,

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(iv) $\left|\Delta(t\eta)-\Delta\left(\left(t-\frac{\pi}{R}\right)\eta\right)\right| \leq CR^{-1}, \qquad t \geq \frac{\pi}{R}.$

Lemma 2 Let $g_0 \in G$, Then

(i) $|S(g_0;t)| = |S(e;t)| = \int_{|\eta|=1} |\Delta(t\eta)|^2 t^{l-1} d\sigma(\eta)$,

(ii) $f_{g_0}(t) = |S(g_0;t)|^{-1} \int_{|\eta|=1} \psi_{g_0}(t\eta) |\Delta(t\eta)|^2 t^{l-1} d\sigma(\eta)$,

where $l = \dim T$ and $\psi_{g_0}(h) = \int_{G/T} f(g_0 g (\exp h) g^{-1}) d\bar{g}$, $h \in H$.

Lemma 3 Let $f \in C(G)$. Then $S_R(f;g)$ converges uniformly to $f(g)$ if and only if there exists a positive number δ such that

$$\int_{\frac{\pi}{R}}^{\delta} \{f_g(t)-f(g)\} t^{-1} \cos\left(Rt-\frac{n\pi}{2}\right) dt = o(1), \quad R \to \infty$$

holds uniformly for $g \in G$.

Lemma 4 Let $f \in C(G)$. Then $\tilde{\omega}(f;t) \leq C\omega(f;t)$.

Referenoes

[1] Голубов, Б. И., О сходимости сферических средних Рисса кратных рядов Фурье, Матем. Сб., 96 (1975), 189—211.

[2] T. I. Akhobadze, On convergence and summability of Fourier series, Anal. Math., 8 (1982), 79—101.

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